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Reply to da Rocha & Rodrigues' comments on the orientation congruent algebra & twisted forms in electrodynamics

Diane G. Demers *

Lansing, Michigan, U.S.A.

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The recent claim by da Rocha and Rodrigues that the orientation congruent algebra ($\mathcal{OC}$ algebra), and by implication the attendant native Clifford algebra, are incompatible with the Clifford bundle approach is false. The native Clifford bundle approach, in fact, *subsumes* the ordinary Clifford bundle one. Associativity is an unnecessarily too strong a requirement for physical applications. As a byproduct of these facts we obtain a new *principle of nonassociative irrelevance* for determining whether a formula is physically meaningful. In addition, the adoption of formalisms that respect the native representation of twisted (or odd) objects and physical quantities is required for the advancement of mathematics, physics, and engineering because they allow equations to be written in sign-invariant form. This perspective simplifies the analysis of, resolves questions about, and ends needless controversies over the signs, orientations, and parities of physical quantities.

1 Introduction

The authors da Rocha and Rodrigues, in their reply [1] to criticism by Itin, Obukhov, and Hehl [2] of their original paper [3], have made a misstatement about the techniques that I have been developing for the algebra and calculus of twisted (or *odd* à la de Rham) objects such as twisted differential forms and vectors in what I call their *native representation*.

The native representation of a twisted objects respects their inherent decomposition into two transversely-oriented parts: a generalized *geometric sign* and a generalized *geometric measure*. The techniques that respect this naturally endowed structure of twisted objects include the *native exterior and Clifford algebras* and the *native exterior calculus*, which are implemented with the *orientation congruent algebra*. See my webpage [4] for more information and some draft papers.

Here I quote exactly (with the original citation numbering) the claims of da Rocha and Rodrigues [1, p. 2] concerning my work:

> Our critics said that our statement that the Clifford bundle works only with pair forms and could not apply to Physics if there is real need for the use of impair forms is *unsubstantied*. They justify their assertion quoting Demers [5] which deals with a non associative 'Clifford like' algebra structure involving pair and impair forms. This structure has nothing to do with the Clifford algebra used (as fibers) in our Clifford bundle, which is an associative algebra, a property that makes that formalism a very powerful computational tool.

Refuting the above claim that the orientation congruent algebra is incompatible with Clifford bundle approach is an elementary exercise. Showing that the native Clifford algebra is compatible with the Clifford

* E-mail: felicity@freeshell.org, felicity@sdf.lonestar.org

bundle approach is a more subtle one. In the process we will discover the remarkable reason for this compatibility.

The next section undertakes these tasks. It begins with a little background exposition.

2 Refutation of the Incompatibility Claim

A key ingredient in all formalisms that respect the native representation of twisted objects is the *orientation congruent algebra* (abbreviated to *$\mathcal{OC}$ algebra*). The orientation congruent algebra is a *Clifford-like* algebra. Clifford-likeness, in the sense that I use this term, may be defined by considering products of the basis multivectors that are derived from an orthonormal set of basis vectors. Then the Clifford-likeness of some algebra means that the product of two given basis multivectors in that algebra is same as their Clifford product *up to a $\pm$ difference in sign*. In other words, the $\mathcal{OC}$ product is a $\pm$ *sign deformation* (or *mutation*) of the Clifford product.

The earliest use of the term *Clifford-like algebra* that I know of was by A. K. Kwaśniewski in a 1985 paper [5]. He used it in a broader sense for the Clifford algebras (in their maximally-graded presentation) of a general k-form $Q(\lambda x) = \lambda^k Q(x)$ rather than the more common quadratic form $Q(\lambda x) = \lambda^2 Q(x)$. Lounesto [6, pp. 284 f.], as well as Hagmark and Lounesto [7], also used the term *Clifford-like*, but in the same restricted sense that I use it, to describe a sign-deformed Clifford product.

An explicit Clifford-like formula for the $\mathcal{OC}$ product is provided in my draft paper [8, pp. 17 ff.]. This formula determines the *sign factor* $\sigma = \pm 1$, that, when applied to the Clifford product, converts it to the orientation congruent product. This sign factor may be expressed as a function of the degrees (or *grades* in Clifford algebra jargon) of the multiplier, multiplicand, and the result of a given orientation congruent multiplication.

Now the sign difference between the Clifford and orientation congruent products may, of course, influence other important properties of these algebras such as whether they are associative or nonassociative. The $\mathcal{OC}$ algebra is indeed a nonassociative algebra (more specifically, a *noncommutative Jordan algebra*). The Clifford algebra, on the other hand, is an associative one.

However, this difference is no impediment to the integration of the $\mathcal{OC}$ algebra *per se* into the Clifford bundle formalism used by da Rocha and Rodrigues [3], [9]. That is because the result of a calculation can always be interconverted between the two products by carrying along with the result of, say, a Clifford multiplication the sign factor σ that transforms it into the result of an orientation congruent multiplication (and vice-versa).

The $\mathcal{OC}$ algebra is used to define both the native exterior algebra and the native Clifford algebra. To understand the remaining issue, how and why the native Clifford and ordinary Clifford algebras (and their bundle-theoretic extensions) are compatible, we need to discuss the orientation congruent algebra's role in the calculation of native Clifford products.

Consider the native representation of twisted objects mentioned in the Introduction. Let us call the generalized geometric sign, the *g-sign*, and the generalized geometric measure, the *gauge*.

The g-sign has properties similar to the sign of a real number, while the gauge has properties similar to the absolute value of a real number. To multiply two real numbers that are decomposed into signs and absolute values, we simply separately multiply their corresponding decomposed parts.

The multiplication of two *geometric* quantities in the native Clifford algebra is exactly analogous to the multiplication of two decomposed real numbers, even to the extent that the products used to multiply the two parts are the same. The surprising fact is that one uses the $\mathcal{OC}$ product to multiply the g-signs of those quantities as well as their gauges. (Although, since the sign of the gauge is irrelevant, one could just as validly use the Clifford product for gauge multiplication.)

Remarkably then the native Clifford algebra is nonassociative—as it must be since the $\mathcal{OC}$ product determines the g-sign. Furthermore, and also quite remarkably, this nonassociativity is irrelevant in all physically (and geometrically) meaningful calculations. Associativity, although extremely convenient in calculations, is an unnecessarily too strong a requirement for physical applications. By implication, we now have a new *principle of nonassociative irrelevance* for determining whether a formula is physically meaningful.

These are the key results that the authors da Rocha and Rodrigues did not know when they claimed that the orientation congruent algebra is incompatible with the Clifford bundle approach. The native Clifford bundle approach, in fact, *subsumes* the ordinary Clifford bundle one.

In the next section I briefly present some reasons why these new techniques deserve serious consideration.

3 The Value of Respecting the Native Structure of Twisted Objects

The authors da Rocha and Rodrigues are fond of quoting [1] [3] a passage from G. de Rham's book *Variétés Différentiables* [10] which we here reproduce in English translation as excerpted from [11]:

> If the manifold V is oriented, that is, if it is orientable and if we have chosen an orientation ε, we may associate with each odd form α an even form $\varepsilon\alpha$. Therefore, in the case of an orientable manifold, we are able to eliminate the use of odd forms by fixing an orientation on V. But for non-orientable manifolds, this concept is in fact useful and natural.

Although it seems that, as de Rham states, one could use a given orientation on an orientable manifold to eliminate odd forms, it turns out that in some problematic cases one may be advised not to do so. And even if one retains odd forms in their original state difficulties may still arise. That is because without the formalisms of the native exterior algebra and the native exterior calculus one is forced to conduct an ad hoc analysis.

The techniques of this framework "automate" what would otherwise be such an ad hoc analysis. Without them, if one does not proceed in the most careful and thoughtful way, one is in danger of making arbitrary and inconsistent sign choices. Furthermore, even if one's ad hoc analysis is conducted correctly, the results may seem capricious and lead to little insight. However, the above mentioned formalisms that I am developing eliminate these difficulties.

Mathematicians, and later and physicists and engineers, have learned to appreciate the unity that Cartan's exterior calculus and derivative brought to the confusing hodge-podge of operators that belong to the Gibbs-Heaviside vector calculus. In a similar way that the exterior calculus is an advancement that allows the equations of electrodynamics to be written in an elegant *operator-invariant* form the native exterior calculus is the next advancement that allows these same equations to also be written in a completely *sign-invariant* form. While many physicists (and even some mathematicians) may scoff by saying that sign and orientation errors are insignificant, I maintain that these techniques that treat signs and orientations rationally and coherently are an important and necessary addition to our formal tool box.

A particularly problematic case occurs if even and odd differential forms (or those even differential forms that were once odd before conversion to even ones) are pulled back between manifolds whose dimensions differ by an odd integer. Two physically relevant examples of this case are (1) the discontinuous electromagnetic boundary conditions (the so-called *jump conditions*) and (2) the apparently inconsistent parities of electromagnetic quantities due to space-time vs. space orientations.

About fifteen years ago example (1) puzzled both Burke [12], [13, p. 332, footnote], and the group of authors Warnick, Selfridge, and Arnold [13]. (See also the confusing discussion of Warnick and Russer [14, pp. 162 f.].) A researcher who uses Burke's modification of exterior calculus to include odd forms [15], [16], as these authors do, in an analysis of the jump conditions is forced to make one of two ad hoc modifications: alter the natural orientation rule either for pullback or for integration. Burke chose the first modification, while Warnick et al. chose the second. However, neither is necessary with the native exterior calculus.

Similar to Cartan's exterior calculus, the techniques of Clifford algebra and geometric calculus (pioneered by Hestenes and Sobczyk [17] and further advanced by Rodrigues and de Oliveira [9]) provide an excellent coordinate-free formalism when a metric is naturally at hand. However, without the techniques of the native Clifford algebra, the author Puska, who analyzes the jump conditions in [18] with the usual Clifford algebra formalism, is unable to write his eqs. (15) and (16) in sign-invariant form.

Example (2) is treated in the paper of the authors Kurz, Auchmann, and Flemisch [19]. Their otherwise excellent analysis is, however, cluttered with some sign-determining factors that could be eliminated with native representation techniques.

Example (2) also occurs in the work of Marmo, Parasecoli, and Tulczyjew [20] (see also [21]). These authors take pains to do a careful analysis of the parities of electromagnetic quantities in space-time and split space+time, but it costs them many pages of calculations. However, with a deep understanding of the underlying principles of orientation and the native formalisms most adapted to them at hand, these calculations can be greatly, but meaningfully, shortened.

In conclusion, unnecessary complexity, perplexing questions, controversies, and misstatements, have been found in the past, and will continue to be found in the future in the mathematical, physical, and engineering literature unless these new techniques that respect the native representation of twisted objects are widely adopted.

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