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STRUCTURALLY-HYPERBOLIC ALGEBRAS DUAL TO THE CAYLEY-DICKSON AND CLIFFORD ALGEBRAS

OR NESTED SNAKES BITE THEIR TAILS

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For Elaine Yaw in honor of friendship

Abstract. The imaginary unit i of $\mathbb{C}$, the complex numbers, squares to -1; while the imaginary unit j of $\mathbb{D}$, the *double number*s (also called *dual* or *split complex numbers*), squares to $+1$. L.H. Kauffman expresses the double number product in terms of the complex number product and vice-versa with two, formally identical, dualizing formulas. The usual sequence of (structurally-elliptic) Cayley-Dickson algebras is $\mathbb{R}, \mathbb{C}, \mathbb{H}, \ldots$, of which Hamilton's quaternions $\mathbb{H}$ generalize to the split quaternions $\mathbb{N}$. Kauffman's expressions are the key to recursively defining the dual sequence of structurally-hyperbolic Cayley-Dickson algebras, $\mathbb{R}, \mathbb{D}, \mathbb{M}, \ldots$, of which Macfarlane's hyperbolic quaternions $\mathbb{M}$ generalize to the split hyperbolic quaternions $\mathbb{M}$. Previously, the structurally-hyperbolic Cayley-Dickson algebras were defined by simply inverting the signs of the squares of the imaginary units of the structurally-elliptic Cayley-Dickson algebras from -1 to $+1$. Using the dual algebras $\mathbb{C}$, $\mathbb{D}$, $\mathbb{H}$, $\mathbb{N}$, $\mathbb{M}$, $\mathbb{M}$, and their further generalizations, we classify the Clifford algebras and their dual *orientation congruent* algebras (Clifford-like, noncommutative Jordan algebras with physical applications) by their representations as tensor products of algebras.

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Contents

1. Introduction

In this paper we develop, for the first time, the ungeneralized, generalized, and ultrageneralized, structurally-hyperbolic Cayley-Dickson algebras *directly as* Cayley-Dickson algebras, that is, in terms of a recursive Cayley-Dickson type product formula. We also describe the representations of the *orientation congruent algebras* $\mathcal{OC}_{p,q}$ by the tensor products of the ultrageneralized, structurally-hyperbolic Cayley-Dickson algebras of two and four dimensions. These representations are dual to the standard representations of the Clifford algebras $\mathcal{C}\ell_{p,q}$ by the tensor products of the generalized (structurally-elliptic) Cayley-Dickson algebras of two and four dimensions, namely, the complex numbers $\mathbb{C}$, the (ungeneralized) quaternions $\mathbb{H}$, the double numbers $\mathbb{D}$ (under the guise of the double rings $\mathbb{R} \oplus \mathbb{R}$ and $\mathbb{H} \oplus \mathbb{H}$), and the split quaternions (under the guise of algebras of $2^{2m} \times 2^{2m}$ matrices, $m \geq 0$, over the algebras of the double rings $\mathbb{R}\oplus\mathbb{R}$ and $\mathbb{H}\oplus\mathbb{H}$), [24, Ch. 16], [33], [34], [35, Ch. 15], [36]. However, while just two (structurally-elliptic) quaternion algebras are sufficient as factor algebras to represent the Clifford algebras, several more ultrageneralized structurally-hyperbolic quaternion algebras are required as factor algebras to represent the orientation congruent algebras.

The orientation congruent (OC) algebras are Clifford-like, noncommutative Jordan algebras that are ingredients in the fundamental reformulation of differential geometry. However, the use of the orientation congruent algebras becomes necessary only when twisted differential forms are pulled back between manifolds whose dimensions differ by an odd integer. The required reformulation of Cartan's exterior calculus in terms of OC algebras resolves some minor dilemmas or controversies that have been reported in the engineering [54, p. 332, fn.] and physics literature G. Marmo et al. [28, 29, 49]. These difficulties involve certain sign ambiguities or discrepancies that are due to the conflicting orientations assigned to twisted differential forms by the usual orientation rules for pull back and integration. However, the details of this application are beyond the scope of this paper.

The ungeneralized structurally-hyperbolic Cayley-Dickson algebras are *hyperbolic* because their (i.e., standard basis vectors other than 1) square to $+1$, in contradistinction to the imaginary units of the traditional Cayley-Dickson algebras which square to -1. As such we refer to the traditional Cayley-Dickson algebras as *structurally-elliptic*. Our use of the term *hyperbolic* is natural, but archaic, since it dates to Alexander Macfarlane (1851-1913) in 1900 [26, 7, 9, 55]. Unfortunately, as discussed in detail shortly, this usage conflicts with another standard one, which is one of two reasons why we have added the modifier *structurally*.

Similarly, the orientation congruent algebras $\mathcal{OC}_n$ of a Euclidean bilinear form (or metric) are *hyperbolic* because under their products the members of an orthonormal set of standard basis vectors for the underlying vector space square to $+1$, in contradistinction to the dually defined Clifford algebras $\mathcal{C}\ell_{0,n}$ under whose products these basis vectors square to -1. In addition, and even more critically, under the product of any orientation congruent algebra $\mathcal{OC}_n$ **all** *basis blades* (i.e., the 2^n elements of the basis of $\mathcal{OC}_n$ as a linear space that are among the exterior products of all combinations of standard basis vectors and 1) square to $+1$, in contradistinction to their behavior under the product of any Clifford algebra $\mathcal{C}\ell_{0,n}$ where, in general, they may square to either $+1$ or -1. This further distinction is described by the use of the modifier *structurally*.

Both the orientation congruent and the Clifford algebras are Clifford-like, the Clifford algebras trivially so. We define an algebra $\mathfrak{A}$ to be *Clifford-like* if all $\mathfrak{A}$-products of basis blades differ by at most a sign from those of some Clifford algebras of a nondegenerate quadratic form. In general, an OC algebra of some nondegenerate quadratic form over $\mathbb{R}$ is *not isomorphic to* some Clifford algebra of a general nondegenerate quadratic form over $\mathbb{R}$. This is evidenced by their distinction in terms of the signs of products of basis blades described in the previous paragraph. Although, the orientation congruent and Clifford algebras differ in their basic definitions and are not, in general, isomorphic when defined over $\mathbb{R}$, it is unclear whether, in general, an OC algebra of some nondegenerate quadratic form over $\mathbb{R}$ is also not isomorphic to some Clifford algebra of a general nondegenerate quadratic form over a *general field*.

Further motivation for the use of the modifier *structurally* arises from the possible confusion over the use of the term *hyperbolic* by other contributors to the mathematical and physical literature to describe the structurally-elliptic Cayley-Dickson algebras that in this paper we prefer to call *split*. Our usage of the term *split* in this way is in accordance with previous usage [59, 58]. The split structurally-elliptic and -hyperbolic CD algebras are defined shortly below.

Now that we have emphasized the distinction between these new systems and the traditional ones by employing the modifier *structurally*, we will generally stop using the cumbersome full phrases *structurally-elliptic* and *structurally-hyperbolic* in favor of the shorter terms *elliptic* and *hyperbolic*. Throughout this paper we may also abbreviate the phrase *Cayley-Dickson* to *CD*. Thus, in particular, the phrases *elliptic Cayley-Dickson* and *hyperbolic Cayley-Dickson* may be shortened to *elliptic CD* and *hyperbolic CD*, respectively.

Having discussed the elliptic and hyperbolic Cayley-Dickson algebras in terms of their multiplication tables, we now turn to their more fundamental definition in terms of a sequence of algebras with a kind of Cayley-Dickson recursive product formula. Both the usual elliptic Cayley-Dickson algebras and the new hyperbolic ones may be constructed by a doubling process that is identical up to the last step of specifying the formula for the product. The product formula for the elliptic Cayley-Dickson algebras is given shortly below. However, a description of the product formula for the hyperbolic Cayley-Dickson algebras is deferred to Section 5.

Both the elliptic and hyperbolic Cayley-Dickson algebra sequences start at level 0 with the real number field $\mathbb{R}$. (In a more general formulation than is required in this introductory paper, the level 0 algebra can be any nonassociative ring [48, p. 9].) Then the numbers of a higher level algebra of either sequence are constructed as ordered pairs of numbers from the algebra just one level below it.

Addition of these pairs is defined as the usual componentwise addition in which the first (resp., second) component of the sum is the sum of the first (resp., second) components of the summands. Multiplication of these pairs is defined by a product formula that is valid for all levels of the construction.

The product formula uses the operation of *conjugation* which is written with an overline as in $\overline{y}$. Conjugation is an anti-involution, meaning that $\overline{y \pm z} = \overline{y} \pm \overline{z}$, $\overline{\overline{y}} = y$ and $\overline{y * z} = \overline{z} * \overline{y}$. At level 0, conjugation is defined as the identity operation: $\overline{y} := y$, for all $y \in \mathbb{R}$. At higher levels, conjugation is defined recursively by the equation $\overline{(w, x)} := (\overline{w}, -x)$, in which the conjugation operation on the right hand side is that of the algebra one level lower than that of the left hand side. The above description of conjugation is based on that of Warren D. Smith [48, p. 9] and Pertti Lounesto [24, pp. 285]

The definition of the ungeneralized CD algebras admits a natural generalization by introducing a parameter $\gamma \in \mathbb{R}$ into the product formula [24, p. 285], [47, pp. 93 f., 201]. The number of such parameters required to define a generalized CD algebra at level n is then just n. Note that the product formula for the ungeneralized elliptic Cayley-Dickson algebras (and the generalized version derived form it) may be expressed in several different, but equivalent, ways [56, fn. 1]; the formula we present next is due to Lounesto. The product in the generalized elliptic case is defined by

Definition 1.1 (The Generalized Elliptic Cayley-Dickson Algebra Product Formula [24, p. 285])**.**

$$(w_1, x_1) * (w_2, x_2) := \big(w_1 w_2 + \gamma \overline{x}_2 x_1, x_2 w_1 + x_1 \overline{w}_2\big). \tag{1.1}$$

Any generalized n-level elliptic Cayley-Dickson algebra may be written in the form $ECD(\gamma_1, \gamma_2, \ldots \gamma_n)$ since it is completely specified by the above Equation (1.1) and the values of the n parameters [24, p. 285]. In the elliptic case and up to the octonions at level 3, we have that $\mathrm{i}^2 = j^1 = \gamma_1$, $j^2 = \gamma_2$, $k^2 = -\gamma_1\gamma_2$, and $l^2 = \gamma_3$. As fully described later in this paper, in the dual hyperbolic case, these equations become $\mathsf{j}^2 = \mathbb{j}^1 = \gamma_1$, $\mathbb{j}^2 = \gamma_2$, $\mathbb{k}^2 = \gamma_1\gamma_2$, and $\mathbb{l}^2 = \gamma_3$.

For the generalized elliptic n-level Cayley-Dickson algebras, when $\gamma_i = -1$ for all $i = 1, \ldots, n$ we recover the ungeneralized elliptic n-level CD algebras from this definition. Dually, for the generalized hyperbolic n-level Cayley-Dickson algebras as defined later, when $\gamma_i = +1$ for all $i = 1, \ldots, n$ we recover the ungeneralized hyperbolic n-level CD algebras.

A couple of standard choices for the parameter γ, which are dual to each other, occur so frequently in this paper that they merit their own definition and description as *split*.

Definition 1.2 (The Split Elliptic and Hyperbolic Cayley-Dickson Algebras)**.** We define the *split elliptic Cayley-Dickson algebras* to be those generalized elliptic Cayley-Dickson algebras generated by setting the highest level parameter γ_n in an n-level algebra to be $+1$ and all other parameters to be -1, as well as all generalized elliptic Cayley-Dickson algebras isomorphic to them. Dually, we define the *split hyperbolic Cayley-Dickson algebras* to be those generalized hyperbolic Cayley-Dickson algebras generated by setting the highest level parameter γ_n in an n-level algebra to be -1 and all other parameters to be $+1$, as well as all generalized hyperbolic Cayley-Dickson algebras isomorphic to them.

As shown in Table 2 of D. Lambert and M. Kibler's paper [22, p. 313], all choices other than $\gamma_1 = \gamma_2 = -1$ for the generalized elliptic quaternions and $\gamma_1 = \gamma_2 = \gamma_3 = -1$ for the generalized elliptic octonions give algebras isomorphic to the split elliptic quaternions and split elliptic octonions, respectively.

In this paper we explicitly construct the first three levels of elliptic Cayley-Dickson algebras and their hyperbolic counterparts, all in both *ungeneralized* and *split* form, and the fourth level in only its ungeneralized form. The names and symbols we use for them are given in the following Table 1.1.

As mentioned above, in the elliptic case, if the γ parameters are all chosen in the standard way to be -1, we get the traditional Cayley-Dickson algebras which in our language are the (ungeneralized) elliptic Cayley-Dickson algebras. The first four of these are the *real number field* $\mathbb{R}$, the *complex number field* $\mathbb{C} = ECD(-1)$ in Table 1.2, the *elliptic quaternion ring* $\mathbb{H} = ECD(-1, -1)$ of Sir William Rowan Hamilton (1805–1865) in Table 1.4, and the *elliptic octonion ring* (or *elliptic octaves*) $\mathbb{O} = ECD(-1, -1, -1)$ of Arthur Cayley (1821–1895) and John Thomas Graves (1806–1870) in Table 1.8.

Also in the elliptic case, if the parameters are all chosen in the way described three paragraphs above, we get the split Cayley-Dickson algebras. The first four of the split Cayley-Dickson algebras are the *real number field* $\mathbb{R}$, the *split complex number ring* $\bcancel{\mathbb{C}} = ECD(+1)$ which is isomorphic to the double number ring $\mathbb{D}$ in Table 1.3, the *split quaternion ring* $\bcancel{\mathbb{H}} = ECD(-1, +1)$ in Table 1.6, and the *split octonion ring* (or *octaves*) $\bcancel{\mathbb{O}} = ECD(-1, -1, +1)$ (not shown).

As mentioned above, dually, in the hyperbolic case, if the γ parameters are all chosen in the standard way to be $+1$, we get the (ungeneralized) hyperbolic Cayley-Dickson algebras. The first four of these are the *real number field* $\mathbb{R}$, the *double number ring* $\mathbb{D} = HCD(+1)$ in Table 1.3, the *hyperbolic quaternion ring* $\mathbb{M} = HCD(+1, +1)$ of Alexander Macfarlane (1851–1913) in Table 1.5, and the *hyperbolic octonion ring* $\mathbb{G} = HCD(+1, +1, +1)$ in Table 1.9.

Also, dually, in the hyperbolic case, if the parameters are all chosen in the way described five paragraphs above, we get the split hyperbolic Cayley-Dickson algebras. The first four of the split hyperbolic Cayley-Dickson algebras are the *real number field* $\mathbb{R}$, the *split double number field* $\bcancel{\mathbb{D}} = HCD(-1)$ which is isomorphic to the complex number algebra $\mathbb{C}$ in Table 1.2, the *split hyperbolic quaternion ring* $\bcancel{\mathbb{M}} = HCD(+1, -1)$ in Table 1.7, and the *split hyperbolic octonion ring* $\bcancel{\mathbb{G}} = HCD(+1, +1, -1)$ (not shown).

TABLE 1.1. The names and symbols used in this paper for the first four levels of the generalized elliptic and hyperbolic Cayley-Dickson algebras.

	Elliptic CD Algebras		Hyperbolic CD Algebras	
Level	Name	Sym.	Name	Sym.
0	real numbers	$\mathbb{R}$	real numbers	$\mathbb{R}$
1	complex numbers	$\mathbb{C}$	double numbers	$\mathbb{D}$
1	split complex numbers	$\bcancel{\mathbb{C}}$	split double numbers	$\bcancel{\mathbb{D}}$
2	elliptic quaternions	$\mathbb{H}$	hyperbolic quaternions	$\mathbb{M}$
2	split elliptic quaternions	$\bcancel{\mathbb{H}}$	split hyperbolic quaternions	$\bcancel{\mathbb{M}}$
3	elliptic octonions	$\mathbb{O}$	hyperbolic octonions	$\mathbb{G}$
3	split elliptic octonions	$\bcancel{\mathbb{O}}$	split hyperbolic octonions	$\bcancel{\mathbb{G}}$

In this and the next two paragraphs we discuss our choice of names and symbols for the algebras listed in Table 1.1. The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are very standard in the literature. The symbol $\mathbb{D}$ for the ring of double numbers is also frequently found in publications. According to the survey paper by Vasile Cruceanu, Pedro Fortuny Ayuso, and Pedro Martínez Gadea [5, p. 87], the double numbers were first described by John Thomas Graves in 1845 [12]. Although, the name *double numbers* is one of many (including *hyperbolic complex numbers*) that appear in the literature [57]. In particular, in applications to differential geometry, the double numbers are also known as the *biparacomplex* [3] or *paracomplex numbers* [5, p. 87].

The designation *double numbers* is used as one of two alternatives in a book of the Russian-American mathematician Boris A. Rosenfeld [40, p. 30] and as the standard in the English translations of the Russian language works of I. M. Yaglom [60, 61]. Following Yaglom, we have chosen the name *double numbers* as our standard.

The designation *split-complex numbers* is also used for $\mathbb{D}$ in some books by Maks A. Akivis and Boris A. Rosenfeld [1, p. 66] [39, pp. 363, 396, 399] [40, p. 30]. Later, in Section 6, to emphasize the dual relationships between the algebra tensor product representations of the Clifford and orientation congruent algebras, we also apply the name *split complex numbers* and our own symbol, the backslashed $\backslash\!\!\!\!\mathbb{C}$ to the double number ring $\mathbb{D}$. In general, throughout this paper, superimposition of a backslash on some symbol for a Cayley-Dickson algebra indicates the split form of that algebra.

In the context of his work, Kauffman [15, 16] very naturally calls $\mathbb{D}$ the *dual numbers*. He explores many interesting dualities between these two number systems that are more subtle than simply having imaginary units with oppositely signed squares, but which stem from this fact. However, we will stay with the terminology of Rosenfeld and Yaglom who apply the name *dual numbers* to the ring of hypercomplex numbers with imaginary unit ϵ such that $\epsilon^2 = 0$.

The symbol $\mathbb{H}$ was chosen, of course, after the name of Sir William Rowan Hamilton, who first published this system. Similarly, it is appropriate to choose $\mathbb{M}$ as the symbol for the hyperbolic quaternions after Alexander Macfarlane who seems to have coined the term as the first to publicly announce his research in 1900 [26]. (For related, earlier work by Macfarlane see [25]).

The *split elliptic quaternions* $\backslash\!\!\!\!\mathbb{H}$ were first described by James Cockle (1819-1895) [8] in 1849 under the name *coquaternions* [8, 59]. In applications to differential geometry, they are also known as the *paraquaternions* [14, p. 222] or *para-quaternions* [6, 59]. In the physics literature, the term *Gödel quaternions* is common [22, p. 312]. This physics terminology is apparently derived from Kurt Gödel's application of the split elliptic quaternions (under the name *hyperbolic quaternions*) to cosmology [11, p. 1416, fn. 14]. Note that there exists a ring isomorphism $\backslash\!\!\!\!\mathbb{H} \cong \mathrm{Mat}(\mathbb{R}, 2)$ [59].

I more arbitrarily designate $\mathbb{G}$ as the symbol for the hyperbolic octonions to honor the same John Thomas Graves mentioned earlier as the first to publish on the double numbers. He was also the first to discover the *elliptic* octonions, but not the first to publish [2, pp. 146 f.] [23, p. 5]. This was due to the Hamilton's tardiness in communicating Graves' results to a journal. Unfortunately, since Arthur Cayley's paper [4] on the subject was thus published before Graves' [12], the elliptic octonions are frequently called the *Cayley numbers* in the literature [2, pp. 146 f.]. Therefore, to give some due to Graves, who missed his chance with the elliptic octonions, I use $\mathbb{G}$ as the symbol for the hyperbolic octonions.

In both the elliptic and hyperbolic cases noncommuting elements first appear at level 2 of the CD algebras with the quaternions. However, while nonassociative elements first appear at level 3 of the elliptic CD algebras with the elliptic octonions, they appear earlier at level 2 of the hyperbolic CD algebras with the hyperbolic quaternions.

Macfarlane discussed the hyperbolic quaternions as an algebra for analyzing the geometry of the two-sheeted hyperboloid just as the elliptic quaternions may be used to analyze the geometry of the sphere [26, 7, 9, 55]. However, his original motivation for investigating them appears to be to compare and reconcile the differences between the then competing systems of algebra and calculus based on Gibbs' vectors vs. Hamilton's quaternions.

My motivation for investigating the hyperbolic quaternions stems from their natural relationship to the orientation congruent algebra. This relationship parallels to some extent that of the elliptic quaternions to the Clifford algebra. For example, the algebra $\mathbb{M}$ is isomorphic to the orientation congruent algebras $\mathcal{OC}_2$ just as the algebra $\mathbb{H}$ is isomorphic to the Clifford algebra $\mathcal{C}\ell_{0,2}$. Also note that the ring $\mathbb{D}$ is isomorphic to both $\mathcal{OC}_1$ and $\mathcal{C}\ell_1$, while the field $\mathbb{C}$ is isomorphic to both $\mathcal{OC}_{0,1}$ and $\mathcal{C}\ell_{0,1}$.

The relationships mentioned in the last paragraph are reflections of another parallel between the relationships of the hyperbolic CD algebras to the orientation congruent algebras and the elliptic CD algebras to Clifford algebras. Namely, both the ungeneralized hyperbolic quaternions and split hyperbolic quaternions play a key role in constructing algebra tensor product representations for the orientation congruent algebras of various signatures just as both the ungeneralized elliptic quaternions and split elliptic quaternions do for the Clifford algebras. Although, the role of the split elliptic quaternions is disguised in the usual presentations by appearing as $2^{2m} \times 2^{2m}$ matrices for nonnegative m with entries from $\mathbb{C}$, $\mathbb{R}$, ${}^2\mathbb{R} = \mathbb{R} \oplus \mathbb{R} \cong \mathbb{D}$, $\mathbb{H}$, or ${}^2\mathbb{H} = \mathbb{H} \oplus \mathbb{H} =\cong \mathbb{D} \otimes \mathbb{H}$, [24, Ch. 16], [33], [34], [35, Ch. 15], [36]. The double and complex numbers are common to the representation theory of both the orientation congruent and Clifford algebras as tensor products of algebras. However, although the factorization of the Clifford algebras into tensor products of algebras requires only the four component algebras $\mathbb{C}$, $\mathbb{D}$, $\mathbb{H}$, and $\cancel{\mathbb{H}}$, the factorization of the orientation congruent algebras requires several even more generalized (*ultrageneralized*) hyperbolic quaternion algebras. This is discussed further in Section 5.

In this paper I make a first attempt to describe some matrix representations for the orientation congruent algebras. These matrix representations are based on the factorization of the OC algebras into tensor products of algebras. Of course, the product used with these matrices cannot be the usual associative one. We investigate the matrix representation of one OC algebra later in Section 4.

The analogy between the relationship of the hyperbolic CD algebras to the orientation congruent algebras and the relationship of the elliptic CD algebras to the Clifford algebras also extends to the next higher level, namely, that of the octonions. At this level an algebra isomorphism is not possible, but the elliptic octonion product may be calculated in terms of the Clifford product. In chapter 23 of his book [24], Lounesto gives three ways to do this, one of which has no parallel in the hyperbolic case, but the other two do. Unfortunately, I do not have time to elaborate here.

There are at least three physical applications of the algebra of Macfarlane's hyperbolic quaternions. The first physical application is to special relativity where the algebra of hyperbolic quaternions may be considered the natural generalization to four dimensions of the double numbers (which nicely formalize the Lorentz transformations two-dimensional spacetime [37, 38]). Thus, the complete group of four-dimensional Lorentz transformations may be expressed in terms of the hyperbolic quaternions. (Although, because of the nonassociativity of $\mathbb{M}$, Joe Rooney rejected this application [38, p. 436].) Hints of how to do this are found in the webpages [9] and [55]. However, I have yet to find the explicit formulas in any source, formal or informal, and so have had to work them out myself. It turns out that certain interesting algebraic twists are necessitated by the nonassociativity of $\mathbb{M}$.

An intriguing reciprocal relationship also exists between the roles of the unit imaginary number i when the elliptic vs. the hyperbolic quaternions are used to express the Lorentz transformations. In both systems i is taken as commuting with all other quantities. (Thus, this i cannot be identified with the quaternions i, j, or k, although these three numbers as well as i square to -1). And, whereas, the elliptic quaternions with real coefficients are naturally suited to describe Euclidean rotations but require an imaginary coefficient to describe boosts, the hyperbolic quaternions with real coefficients are naturally suited to describe boosts and require an imaginary coefficient to describe rotations. Unfortunately, I do not have time to present more details here.

The role of the nonassociative hyperbolic quaternions in special relativity and their relationship to the orientation congruent algebra suggests a possible connection of both these algebras to Abraham A. Ungar's work. Ungar's *gyrogroup* approach to special relativity and hyperbolic geometry derives from his analysis of the essential nonassociativity of the finite Thomas or Wigner rotation. It is expounded in his books [52, 53] (and many papers, not cited here). Specifically, it may be possible to derive the Thomas or Wigner rotation in another, or even more direct way, by using the nonassociative hyperbolic quaternions. However, these connections cannot be pursued further at this time.

The second physical application of the hyperbolic quaternions is to electromagnetism. However, I have only seen this as the abstract [10] of a contribution by the authors Suleyman Demir, Murat Tanişh, and Nuray Candemir to a recently held conference. According to this abstract: "Maxwell's equations and relevant field equations are investigated with hyperbolic quaternions, and these equations have been given in compact, simpler and elegant forms. Derived equations are compared with their vectorial, complex quaternionic, dual quaternionic and octonionic representations, as well."

The third physical application of the hyperbolic quaternions is to quantum mechanics. Macfarlane's algebra $\mathbb{M}$ was used by Dionisios Margetis and Manoussos G. Grillakis in their paper [27] to describe the loss of a pure state in the memory of a quantum computer. However, I have not confirmed the results of these authors by a careful reading of their paper. Some other work on the hyperbolic approach to quantum mechanics that is presented by Stefan Ulrych also appears promising [50, 51]. However, the possible application of the concepts of the present paper to quantum theory awaits further investigation.

The orientation congruent algebra arose itself from an application in mathematical physics for which it has been named. That is, the OC algebra can be used to represent odd (or twisted) differential forms and calculate their exterior products and derivatives. It has long been known that odd differential forms are naturally endowed with the two transversely oriented parts: a *generalized sign* and a *generalized magnitude*. That is clear from the early illustrations in Schouten's books and papers, [43, p. 22], [46], [44, p. 28], and [45, pp. 31–33, 55]. (See also Salgado [41, 42].) But it takes the orientation congruent algebra to realize this graphical truism as an algebraic system.

There are easier ways to algebraically represent odd differential forms. However, the orientation congruent algebra approach is necessary if odd forms are pulled back between manifolds with an odd difference in dimensions. In this case the usual analyses generally lead to a sign discrepancy. Physical examples in which this type of sign confusion arises include the electromagnetic boundary conditions discussed by K. Warnick et al. [54, p. 332, fn.], as well as the apparently inconsistent parities of electromagnetic quantities due to space-time vs. space orientations discussed by G. Marmo et al. [28, 29].

In all these applications the orientation congruent algebra plays a intermediate role in the calculation. Sometimes its nonassociativity is crucial, as when pulling back twisted forms between manifolds whose dimensions differ by an odd integer. On the other hand, sometimes it is not, as when performing a boost which can also be done with the associative matrix product between matrices representing linear transformations.

The following list gives the names and symbols for some of the primary algebras used in this paper along with their standard basis elements.

The complex number field, $\mathbb{C}$**:** $1, \mathsf{i}$
The split complex number ring, $\not{\mathbb{C}}$**:** ($\not{\mathbb{C}}$ is isomorphic to $\mathbb{D}$)
The double number ring, $\mathbb{D}$**:** $1, \mathsf{j}$
The split double number field, $\not{\mathbb{D}}$**:** ($\not{\mathbb{D}}$ is isomorphic to $\mathbb{C}$)
Hamilton's elliptic quaternions, $\mathbb{H}$**:** $1, i, j, k$
The split elliptic quaternions, $\not{\mathbb{H}}$**:** $1, \not{i}, \not{j}, \not{k}$
Macfarlane's hyperbolic quaternions, $\mathbb{M}$**:** $1, \mathbb{i}, \mathbb{j}, \mathbb{k}$
The split hyperbolic quaternions, $\not{\mathbb{M}}$**:** $1, \not{\mathbb{i}}, \not{\mathbb{j}}, \not{\mathbb{k}}$
The elliptic octonions, $\mathbb{O}$**:** $1, i, j, k, l, il, jl, kl$
The split elliptic octonions, $\not{\mathbb{O}}$**:** $1, \not{i}, \not{j}, \not{k}, \not{l}, \not{i}\not{l}, \not{j}\not{l}, \not{k}\not{l}$
The hyperbolic octonions, $\mathbb{G}$**:** $1, \mathbb{i}, \mathbb{j}, \mathbb{k}, \mathbb{l}, \mathbb{i}\mathbb{l}, \mathbb{j}\mathbb{l}, \mathbb{k}\mathbb{l}$
The split hyperbolic octonions, $\not{\mathbb{G}}$**:** $1, \not{\mathbb{i}}, \not{\mathbb{j}}, \not{\mathbb{k}}, \not{\mathbb{l}}, \not{\mathbb{i}}\not{\mathbb{l}}, \not{\mathbb{j}}\not{\mathbb{l}}, \not{\mathbb{k}}\not{\mathbb{l}}$
The Clifford algebras, $\mathcal{C}\ell_{p,q}$**:** $1, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{12}, \mathbf{e}_{31}, \ldots, \mathbf{e}_{12\ldots p+q}$
The orientation congruent algebras, $\mathcal{OC}_{p,q}$**:** $1, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{12}, \mathbf{e}_{31}, \ldots, \mathbf{e}_{12\ldots p+q}$

The multiplication tables for most of these algebras follow. Although these tables take up several pages, one of the best ways to understand an algebra is to examine its multiplication table. And, although the arrangement and choice of some of the standard elements in these tables may be somewhat unconventional, they are very suitable to our investigations.

Table 1.2. The multiplication table for the field $\mathbb{C}$ of the usual complex numbers. The imaginary unit is i. The complex number field is isomorphic to the Clifford algebra $\mathcal{C}\ell_{0,1}$, the orientation congruent algebra $\mathcal{OC}_{0,1}$, and the split double number field $\not{\mathbb{D}}$. The product of $\mathbb{C}$ is symbolized as $*$. It is uncircled to indicate duality with the product symbol $\circledast$ of Table 1.3.

		b	
	$a * b$	1	i
a	1	1	i
	i	i	-1

Table 1.3. The multiplication table for the ring $\mathbb{D}$ of double or split complex numbers. The imaginary unit is j. The double number ring is isomorphic to the Clifford algebra $\mathcal{C}\ell_{1}$, the orientation congruent algebra $\mathcal{OC}_{1}$, and the split complex number ring $\not{\mathbb{C}}$. The product of $\mathbb{D}$ is symbolized as $\circledast$. It is circled to indicate duality with the product symbol $*$ of Table 1.2.

		b	
	$a \circledast b$	1	j
a	1	1	j
	j	j	1

Table 1.4. The multiplication table for the algebra $\mathbb{H}$ of Hamilton's elliptic quaternions. It is isomorphic to the Clifford algebra $\mathcal{C}\ell_{0,2}$. The product of $\mathbb{H}$ is symbolized as $*$. It is uncircled to indicate duality with the product symbol $\circledast$ of Table 1.5.

		b			
	$a * b$	1	i	j	k
a	1	1	i	j	k
	i	i	-1	k	$-j$
	j	j	$-k$	-1	i
	k	k	j	$-i$	-1

Table 1.5. The multiplication table for the algebra $\mathbb{M}$ of Macfarlane's hyperbolic quaternions. It is isomorphic to the orientation congruent algebra $\mathcal{OC}_2$. The product of $\mathbb{M}$ is symbolized as $\circledast$. It is circled to indicate duality with the product symbol $*$ of Table 1.4.

		b			
	$a \circledast b$	1	$\mathbb{i}$	$\mathbb{j}$	$\mathbb{k}$
a	1	1	$\mathbb{i}$	$\mathbb{j}$	$\mathbb{k}$
	$\mathbb{i}$	$\mathbb{i}$	1	$\mathbb{k}$	$-\mathbb{j}$
	$\mathbb{j}$	$\mathbb{j}$	$-\mathbb{k}$	1	$\mathbb{i}$
	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{j}$	$-\mathbb{i}$	1

Table 1.6. The multiplication table for the algebra $\not{\mathbb{H}}$ of split elliptic quaternions. It is isomorphic to the algebras $\mathrm{Mat}(\mathbb{R}, 2)$, $\mathcal{C}\ell_{1,1}$, and $\mathcal{C}\ell_2$. The product of $\not{\mathbb{H}}$ is symbolized as $\not\ast$. It is uncircled to indicate duality with the product symbol $\not\circledast$ of Table 1.7. The products in red cells are signed oppositely to those in Table 1.4 for Hamilton's (non-split) elliptic quaternions.

		b			
	$a \not\ast b$	1	$\not i$	$\not j$	$\not k$
	1	1	$\not i$	$\not j$	$\not k$
	$\not i$	$\not i$	-1	$\not k$	$-\not j$
a	$\not j$	$\not j$	$-\not k$	1	$-\not i$
	$\not k$	$\not k$	$\not j$	$\not i$	1

Table 1.7. The multiplication table for the algebra $\not{\mathbb{M}}$ of split hyperbolic quaternions. It is isomorphic to the algebras $\mathrm{HMat}(\mathbb{D}, 2)$, $\mathcal{OC}_{1,1}$, and $\mathcal{OC}_{0,2}$. The product of $\not{\mathbb{M}}$ is symbolized as $\not\circledast$. It is circled to indicate duality with the product symbol $\not\ast$ of Table 1.6. The products in red cells are signed oppositely to those in Table 1.5 for Macfarlane's (non-split) hyperbolic quaternions.

		b			
	$a \not\circledast b$	1	$\not{\mathbb{i}}$	$\not{\mathbb{j}}$	$\not{\mathbb{k}}$
	1	1	$\not{\mathbb{i}}$	$\not{\mathbb{j}}$	$\not{\mathbb{k}}$
	$\not{\mathbb{i}}$	$\not{\mathbb{i}}$	1	$\not{\mathbb{k}}$	$-\not{\mathbb{j}}$
a	$\not{\mathbb{j}}$	$\not{\mathbb{j}}$	$-\not{\mathbb{k}}$	-1	$-\not{\mathbb{i}}$
	$\not{\mathbb{k}}$	$\not{\mathbb{k}}$	$\not{\mathbb{j}}$	$\not{\mathbb{i}}$	-1

Table 1.8. The multiplication table for the algebra $\mathbb{O}$ of Cayley-Graves elliptic octonions. The factors are in reflected complementary order. The product of $\mathbb{O}$ is symbolized as $*$. It is uncircled to indicate duality with the product symbol $\circledast$ of Table 1.9.

		b							
	$a * b$	1	i	j	k	kl	jl	il	l
a	1	1	i	j	k	kl	jl	il	l
	i	i	-1	k	$-j$	jl	$-kl$	$-l$	il
	j	j	$-k$	-1	i	$-il$	$-l$	kl	jl
	k	k	j	$-i$	-1	$-l$	il	$-jl$	kl
	kl	kl	$-jl$	il	l	-1	i	$-j$	$-k$
	jl	jl	kl	l	$-il$	$-i$	-1	k	$-j$
	il	il	l	$-kl$	jl	j	$-k$	-1	$-i$
	l	l	$-il$	$-jl$	$-kl$	k	j	i	-1

Table 1.9. The multiplication table for the algebra $\mathbb{G}$ of hyperbolic octonions. The factors are in reflected complementary order. The product of $\mathbb{G}$ is symbolized as $\circledast$. It is circled to indicate duality with the product symbol $*$ of Table 1.8.

		b							
	$a \circledast b$	1	$\mathbb{i}$	$\mathbb{j}$	$\mathbb{k}$	$\mathbb{k}\mathbb{l}$	$\mathbb{j}\mathbb{l}$	$\mathbb{i}\mathbb{l}$	$\mathbb{l}$
a	1	1	$\mathbb{i}$	$\mathbb{j}$	$\mathbb{k}$	$\mathbb{k}\mathbb{l}$	$\mathbb{j}\mathbb{l}$	$\mathbb{i}\mathbb{l}$	$\mathbb{l}$
	$\mathbb{i}$	$\mathbb{i}$	1	$\mathbb{k}$	$-\mathbb{j}$	$\mathbb{j}\mathbb{l}$	$-\mathbb{k}\mathbb{l}$	$-\mathbb{l}$	$\mathbb{i}\mathbb{l}$
	$\mathbb{j}$	$\mathbb{j}$	$-\mathbb{k}$	1	$\mathbb{i}$	$-\mathbb{i}\mathbb{l}$	$-\mathbb{l}$	$\mathbb{k}\mathbb{l}$	$\mathbb{j}\mathbb{l}$
	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{j}$	$-\mathbb{i}$	1	$-\mathbb{l}$	$\mathbb{i}\mathbb{l}$	$-\mathbb{j}\mathbb{l}$	$\mathbb{k}\mathbb{l}$
	$\mathbb{k}\mathbb{l}$	$\mathbb{k}\mathbb{l}$	$-\mathbb{j}\mathbb{l}$	$\mathbb{i}\mathbb{l}$	$\mathbb{l}$	1	$\mathbb{i}$	$-\mathbb{j}$	$-\mathbb{k}$
	$\mathbb{j}\mathbb{l}$	$\mathbb{j}\mathbb{l}$	$\mathbb{k}\mathbb{l}$	$\mathbb{l}$	$-\mathbb{i}\mathbb{l}$	$-\mathbb{i}$	1	$\mathbb{k}$	$-\mathbb{j}$
	$\mathbb{i}\mathbb{l}$	$\mathbb{i}\mathbb{l}$	$\mathbb{l}$	$-\mathbb{k}\mathbb{l}$	$\mathbb{j}\mathbb{l}$	$\mathbb{j}$	$-\mathbb{k}$	1	$-\mathbb{i}$
	$\mathbb{l}$	$\mathbb{l}$	$-\mathbb{i}\mathbb{l}$	$-\mathbb{j}\mathbb{l}$	$-\mathbb{k}\mathbb{l}$	$\mathbb{k}$	$\mathbb{j}$	$\mathbb{i}$	1

Table 1.10. The multiplication table for the Clifford algebra $\mathcal{C}\ell_3$. The factors are in reflected, complementary grade order with indices in orientation congruent order.

		b							
	$a \circ b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{I}$
	1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{I}$
	$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$\mathbf{I}$	$\mathbf{e}_{23}$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_1$	$\mathbf{I}$	$\mathbf{e}_3$	$\mathbf{e}_{31}$
a	$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\mathbf{I}$	$\mathbf{e}_1$	$-\mathbf{e}_2$	$\mathbf{e}_{12}$
	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{I}$	-1	$\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_3$
	$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$\mathbf{I}$	$-\mathbf{e}_1$	$-\mathbf{e}_{23}$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_2$
	$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\mathbf{I}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	-1	$-\mathbf{e}_1$
	$\mathbf{I}$	$\mathbf{I}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	-1

Table 1.11. The multiplication table for the orientation congruent algebra $\mathcal{OC}_3$. The factors and indices are ordered as in Table 1.10 above. Red cell entries are signed oppositely to those in Table 1.10.

		b							
	$a \circledcirc b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{\Omega}$
	1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{\Omega}$
	$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{\Omega}$	$\mathbf{e}_{23}$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$\mathbf{e}_1$	$\mathbf{\Omega}$	$-\mathbf{e}_3$	$\mathbf{e}_{31}$
a	$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\mathbf{\Omega}$	$-\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$\mathbf{\Omega}$	1	$-\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$
	$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$-\mathbf{e}_3$	$\mathbf{\Omega}$	$\mathbf{e}_1$	$\mathbf{e}_{23}$	1	$-\mathbf{e}_{12}$	$\mathbf{e}_2$
	$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\mathbf{\Omega}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_{31}$	$\mathbf{e}_{12}$	1	$\mathbf{e}_1$
	$\mathbf{\Omega}$	$\mathbf{\Omega}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_1$	1

2. The Kauffman Product Duality Theorems

> This may seem to be a rather complicated route to $\mathrm{i}^2 = -1$, but it is the point at which the snake bites its tail: As the next lemma shows, each multiplication is expressed in terms of the other by the same formula!
>
> Louis H. Kauffman [15, p. 210]

Some results of Louis H. Kauffman's paper *Complex Numbers and Algebraic Logic* [15] are the key to this section, which, in turn, forms the basis for the next three sections. In this paper, Kauffman presents the two complex number and double number *product duality* formulas, one in his Lemma 2.2 and the other, its dual version, slightly preceding it. These formulas of Kauffman are the first step toward defining the hyperbolic Cayley-Dickson algebras *exactly as* Cayley-Dickson algebras, that is, through a recursive product of number pairs from the next lower level.

In turn, the hyperbolic CD algebras are the key to the matrix representation of the orientation congruent algebra, and also to the duality classification of the Clifford and orientation congruent algebras as dual Clifford-like algebras, the former algebra being elliptic, and the latter, hyperbolic. But that is the subject of Section 6.

Lemma 2.1 below contains Kauffman's original product duality formulas which apply only to the algebras at level 1 of the elliptic and hyperbolic Cayley-Dickson sequences, the complex and dual number algebras, $\mathbb{C}$ and $\mathbb{D}$. Although Kauffman's original formulas apply at level 1 only, in this paper we use the phrase *Kauffman product duality* generically for the analogous relationship of equality between the product of one hyperbolic (elliptic) Cayley-Dickson algebra to an expression written in terms of the product of its dual elliptic (resp., hyperbolic) Cayley-Dickson algebra.

In the Equations (2.1) of Lemma 2.1 Kauffman's original product duality formulas are rewritten in a notation consistent with the conventions of this paper in which, generally, a symbol that is circled indicates a hyperbolic algebra product, while the same symbol without a circle indicates the corresponding elliptic algebra product. Thus, in Equations (2.1), $\alpha \circledast \beta$ is the double number product of α and β with multiplication table, Table 1.3, and $\alpha * \beta$ is the familiar complex number product of α and β with multiplication table, Table 1.2. The overline indicates complex or double number conjugation.

Lemma 2.1 (Duality of the Complex and Double Number Algebras)**.**

$$\alpha \circledast \beta = \frac{1}{2}\left(\alpha * \beta + \overline{\alpha} * \beta + \alpha * \overline{\beta} - \overline{\alpha} * \overline{\beta}\right) \tag{2.1a}$$

$$\alpha * \beta = \frac{1}{2}\left(\alpha \circledast \beta + \overline{\alpha} \circledast \beta + \alpha \circledast \overline{\beta} - \overline{\alpha} \circledast \overline{\beta}\right) \tag{2.1b}$$

Proof. In Table 2.1 we prove Kauffman's original product duality formulas for the complex and double numbers by straightforward calculation. In fact, we prove both formulas, which are duals of each other, simultaneously by writing those signs that vary with the *internal* product, that is, the product on the right hand sides of Equations (2.1), as coordinated plus-or-minus $\pm$ and minus-or-plus $\mp$ signs. Our convention is that the top sign applies to the case in which the internal product is the complex number product, while the bottom sign applies to the case in which it is the double number product.

In this proof, we write the components of a complex or double number γ as an ordered pair $\gamma = (\gamma_0, \gamma_1)$. We use this ambiguous notation purposely so that a given ordered pair may represent either a complex or a double number, $\gamma = \gamma_0 + \mathrm{i}\gamma_1$ or $\gamma_0 + \mathrm{j}\gamma_1$. Also, for the same reason, product symbols are not used in Table 2.1. □

TABLE 2.1. Simultaneous proof of both of Kauffman's original product duality Equations 2.1 [15, p. 219] for level 1 of the Cayley-Dickson sequence: the complex and dual numbers, $\mathbb{C}$ and $\mathbb{D}$. The top sign in the symbols $\pm$ and $\mp$ applies when the *internal* product is the complex number product, and the bottom sign applies when it is the double number product.

The Parts of γ	The Subterms of the Four Terms				$\frac{1}{2}$ of Row Sums
γ_0, γ_1	$+\alpha\beta$	$+\overline{\alpha}\beta$	$+\alpha\overline{\beta}$	$-\overline{\alpha}\overline{\beta}$	$\gamma = \alpha\beta$
γ_0	$+\alpha_0\beta_0$ $\mp\alpha_1\beta_1$	$+\alpha_0\beta_0$ $\pm\alpha_1\beta_1$	$+\alpha_0\beta_0$ $\pm\alpha_1\beta_1$	$-\alpha_0\beta_0$ $\pm\alpha_1\beta_1$	$+\alpha_0\beta_0$ $\pm\alpha_1\beta_1$
γ_1	$+\alpha_0\beta_1$ $+\alpha_1\beta_0$	$+\alpha_0\beta_1$ $-\alpha_1\beta_0$	$-\alpha_0\beta_1$ $+\alpha_1\beta_0$	$+\alpha_0\beta_1$ $+\alpha_1\beta_0$	$+\alpha_0\beta_1$ $+\alpha_1\beta_0$

The proof given in Table 2.1 was only a preliminary to the next tabular proof of another lemma extending the original Kauffman product duality formulas, which apply only to the real, complex, and double numbers, to the next higher level of the elliptic and hyperbolic Cayley-Dickson algebras—the elliptic and hyperbolic quaternions.

Lemma 2.2 (Duality of the Elliptic and Hyperbolic Quaternion Algebras)**.**

$$\alpha \circledast \beta = \frac{1}{2}\left(\beta * \alpha + \overline{\beta} * \alpha + \beta * \overline{\alpha} - \overline{\beta} * \overline{\alpha}\right) \tag{2.2a}$$

$$\alpha * \beta = \frac{1}{2}\left(\beta \circledast \alpha + \overline{\beta} \circledast \alpha + \beta \circledast \overline{\alpha} - \overline{\beta} \circledast \overline{\alpha}\right) \tag{2.2b}$$

Proof. In the proof of Table 2.2 we tabulate only the signs of each subterm in the sum defining a component part of the quaternion $\gamma = \alpha\beta$. Again we write these numbers as an ordered n-tuple of components, where n, in this quaternion case, is 4: $\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$. Here a crucial difference from the complex and double number algebras, $\mathbb{C}$ and $\mathbb{D}$, at level 1 of the Cayley-Dickson sequence, is that the extension of Kauffman's product duality formulas to the elliptic and hyperbolic quaternions, $\mathbb{H}$ and $\mathbb{M}$, at level 2 of the Cayley-Dickson sequence, requires that they be modified by transposing the factors in the product. This gives the so-called *opposite* product. We indicate it in Table 2.1 as γ^{op}. However, this is the opposite of the Kauffman duality product. As the last column shows γ^{op} is actually the desired quaternion product. □

The following theorem gives the Kauffman product duality formulas as extended to all levels of the elliptic and hyperbolic Cayley-Dickson algebra sequences.

Table 2.2. Simultaneous proof of the two Kauffman product duality Equations 2.2 as extended in this paper to the algebras at level 2 of the Cayley-Dickson sequence: the elliptic and hyperbolic quaternions, $\mathbb{H}$ and $\mathbb{M}$. The top sign in the symbols $\pm$ and $\mp$ applies when the *internal* product is the elliptic quaternion product, and the bottom sign applies when it is the hyperbolic quaternion product. Here, for simplicity, we show only the signs of each subterm of the four terms while omitting the symbols for the components themselves.

The Parts of γ	The Subterms of the Four Terms				$\frac{1}{2}$ of Row Sums	
$\gamma_0, \gamma_1, \gamma_2, \gamma_3$	$+\alpha\beta$	$+\overline{\alpha}\beta$	$+\alpha\overline{\beta}$	$-\overline{\alpha}\overline{\beta}$	$\gamma = \alpha\beta$	$\gamma^{\text{op}} = \beta\alpha$
γ_0	$+$	$+$	$+$	$-$	$+\alpha_0\beta_0$	$+\alpha_0\beta_0$
	$\mp$	$\pm$	$\pm$	$\pm$	$\pm\alpha_1\beta_1$	$\pm\alpha_1\beta_1$
	$\mp$	$\pm$	$\pm$	$\pm$	$\pm\alpha_2\beta_2$	$\pm\alpha_2\beta_2$
	$\mp$	$\pm$	$\pm$	$\pm$	$\pm\alpha_3\beta_3$	$\pm\alpha_3\beta_3$
γ_1	$+$	$+$	$-$	$+$	$+\alpha_0\beta_1$	$+\alpha_0\beta_1$
	$+$	$-$	$+$	$+$	$+\alpha_1\beta_0$	$+\alpha_1\beta_0$
	$+$	$-$	$-$	$-$	$-\alpha_2\beta_3$	$+\alpha_2\beta_3$
	$-$	$+$	$+$	$+$	$+\alpha_3\beta_2$	$-\alpha_3\beta_2$
γ_2, γ_3	The γ_2 and γ_3 parts are analogous to the γ_1 one.					

Theorem 2.3 (Duality of All Ungeneralized Elliptic and Hyperbolic CD Algebras)**.**

$$\alpha \circledast \beta = \frac{1}{2}\left(\beta * \alpha + \overline{\beta} * \alpha + \beta * \overline{\alpha} - \overline{\beta} * \overline{\alpha}\right) \tag{2.3a}$$

$$\alpha * \beta = \frac{1}{2}\left(\beta \circledast \alpha + \overline{\beta} \circledast \alpha + \beta \circledast \overline{\alpha} - \overline{\beta} \circledast \overline{\alpha}\right) \tag{2.3b}$$

Proof. The extension beyond levels 0 (the real numbers), 1 (the complex and double numbers), and 2 (the elliptic and hyperbolic quaternions) of the elliptic and hyperbolic Cayley-Dickson algebra sequences is by simple induction from Lemma 2.2. Note that only the noncommutative property of these products (which first arises for the quaternion algebras at level 2) must be explicitly reflected in the product duality formulas. No other algebraic properties, such as the various forms of nonassociativity, need to be explicitly taken into account in these formulas since all other algebraic properties involve products of more than two factors. □

The following Theorem 2.4 gives the Kauffman product duality formulas as extended to the split elliptic and hyperbolic quaternions. We use Equation (2.4a) of this theorem in Section 4 to construct a full matrix algebra representation for the split hyperbolic quaternions $\mathbb{M}$. As symbolized by $\mathrm{HMat}(\mathbb{R},2)$ this matrix representation is a basic component in the matrix representations of the orientation congruent algebras $\mathcal{OC}_{p,q}$. The role of $\mathrm{HMat}(\mathbb{R},2)$ in the matrix representations of the orientation congruent algebras $\mathcal{OC}_{p,q}$ is analogous to that of the full matrix algebra of real 2×2 matrices $\mathrm{Mat}(\mathbb{R},2)$ in the matrix representations of the Clifford algebras $\mathcal{C}\ell_{p,q}$. These matrix representations will be developed later in Section 6.

Theorem 2.4 (Duality of the Split Elliptic and Hyperbolic Quaternions)**.**

$$\alpha \circledast \beta = \frac{1}{2}\left(\beta * \alpha + \overline{\beta} * \alpha + \beta * \overline{\alpha} - \overline{\beta} * \overline{\alpha}\right) \tag{2.4a}$$

$$\alpha * \beta = \frac{1}{2}\left(\beta \circledast \alpha + \overline{\beta} \circledast \alpha + \beta \circledast \overline{\alpha} - \overline{\beta} \circledast \overline{\alpha}\right) \tag{2.4b}$$

Proof. The proof in Table 2.3 is analogous to that of Lemma 2.2. □

In the following we derive the hyperbolic Cayley-Dickson formula. The first step is, using the first extended Kauffman product duality formula, Equation 2.3a, to determine the hyperbolic quaternion matrix representation and its nonassociative matrix product. It turns out that the these matrix representations are complex with entries from $\mathbb{C}$. The entries of these matrices are also linearly dependent so that the nonassociative matrix product actually defines a product of pairs of complex numbers. Then, we use the second extended Kauffman product duality formula, Equation 2.3b, to expand each complex number product occurring in this nonassociative product of complex number pairs into an internal product that is the product of double numbers. This is our desired result—an expression recursively defining the hyperbolic Cayley-Dickson algebras as pairs of numbers from the hyperbolic Cayley-Dickson algebra at the next lower level.

Table 2.3. Simultaneous proof of the two Kauffman product duality Equations 2.4 as extended in this paper to the split algebras at level 2 of the Cayley-Dickson sequence: the split elliptic and hyperbolic quaternions, $\mathbb{H}$ and $\mathbb{M}$. The top sign in the symbols $\pm$ and $\mp$ applies when the *internal* product is the split elliptic quaternion product, and the bottom sign applies when it is the split hyperbolic quaternion product. Here, again for simplicity, we show only the signs of each subterm of the four terms while omitting the symbols for the components themselves.

The Parts of γ	The Subterms of the Four Terms				$\frac{1}{2}$ of Row Sums	
$\gamma_0, \gamma_1, \gamma_2, \gamma_3$	$+\alpha\beta$	$+\overline{\alpha}\beta$	$+\alpha\overline{\beta}$	$-\overline{\alpha}\overline{\beta}$	$\gamma = \alpha\beta$	$\gamma^{\mathrm{op}} = \beta\alpha$
	$+$	$+$	$+$	$-$	$+\alpha_0\beta_0$	$+\alpha_0\beta_0$
γ_0	$\mp$	$\pm$	$\pm$	$\pm$	$\pm\alpha_1\beta_1$	$\pm\alpha_1\beta_1$
	$\pm$	$\mp$	$\mp$	$\mp$	$\mp\alpha_2\beta_2$	$\mp\alpha_2\beta_2$
	$\pm$	$\mp$	$\mp$	$\mp$	$\mp\alpha_3\beta_3$	$\mp\alpha_3\beta_3$
	$+$	$+$	$-$	$+$	$+\alpha_0\beta_1$	$+\alpha_0\beta_1$
γ_1	$+$	$-$	$+$	$+$	$+\alpha_1\beta_0$	$+\alpha_1\beta_0$
	$-$	$+$	$+$	$+$	$+\alpha_2\beta_3$	$-\alpha_2\beta_3$
	$+$	$-$	$-$	$-$	$-\alpha_3\beta_2$	$+\alpha_3\beta_2$
	$+$	$+$	$-$	$+$	$+\alpha_0\beta_2$	$+\alpha_0\beta_2$
γ_2	$+$	$-$	$+$	$+$	$+\alpha_2\beta_0$	$+\alpha_2\beta_0$
	$+$	$-$	$-$	$-$	$-\alpha_3\beta_1$	$+\alpha_3\beta_1$
	$-$	$+$	$+$	$+$	$+\alpha_1\beta_3$	$-\alpha_1\beta_3$
γ_3	The γ_3 part is analogous to the γ_2 one.					

3. Hamilton and Dirac Coordinates

As is well known, the complex numbers may be defined of as ordered pairs of real numbers (u, v) subject to the multiplication rule $(u_1, v_1) * (u_2, v_2) = (u_1u_2 - v_1v_2, u_1v_2 + u_2v_1)$. According to Lounesto [24, p. 31], this formulation is due to Hamilton. Therefore, we use the term *Hamilton coordinates* to refer to this use of ordered pairs to express complex numbers. However, we extend this terminology to refer not only to complex numbers, but also to numbers of any elliptic or hyperbolic CD algebra. We also say that a number is in Hamilton coordinates not only if it expressed as a simple pair of numbers from the next lower algebra, but also if it is expressed as pairs of pairs of numbers from the second lower algebra. Furthermore, we allow such expansions of numbers into pairs of numbers from lower level algebras to be carried as far down as one wishes, but uniformly, so that only numbers of an elliptic or hyperbolic CD algebra from a given level appear anywhere in the expansion. And, of course, these expansions can contain numbers no lower than those in $\mathbb{R}$ at level 0.

The following works of Louis H. Kauffman provide, in part, an introduction to the physical meaning of Dirac coordinates (or *iterants* in Kauffman's terms) in special relativity, *Complex Numbers and Algebraic Logic* [15], *Transformations in special Relativity* [16], *Knot Logic* [17], and *Knots and Physics* [18, pp. 392–402, 460–466]. For more references and a further discussion of the physical significance of Dirac coordinates see the paper by Kim and Noz, *Dirac's Light-Cone Coordinate System* [20].

Table 3.1. The standard basis elements of $\mathbb{D}$, the ring of double numbers, written in Hamilton and Dirac coordinates. In this table and those following the Dirac coordinates are written with the plus sign "+" representing $+1$ and the negative sign "−" representing -1.

Symbol Level 1	Hamilton Coordinates Level 0	Dirac Coordinates Level 0
1	$(1, 0)$	$[+, +]$
j	$(0, 1)$	$[+, -]$

Table 3.2. The standard basis elements of $\mathbb{M}$, the ring of Macfarlane's hyperbolic quaternions, written in Hamilton and Dirac coordinates.

Symbol	Hamilton Coordinates		Dirac Coordinates	
Level 2	Level 1	Level 0	Level 1	Level 0
1	$(1, 0)$	$((1, 0), (0, 0))$	$[1, \quad 1]$	$[[+, +], [+, +]]$
$\mathbb{i}$	$(\mathsf{j}, 0)$	$((0, 1), (0, 0))$	$[\mathsf{j}, \quad \mathsf{j}]$	$[[+, -], [+, -]]$
$\mathbb{j}$	$(0, 1)$	$((0, 0), (1, 0))$	$[1, \ -1]$	$[[+, +], [-, -]]$
$\mathbb{k}$	$(0, \mathsf{j})$	$((0, 0), (0, 1))$	$[\mathsf{j}, \ -\mathsf{j}]$	$[[+, -], [-, +]]$

Note that the sequences consisting of $+1$ and -1 that appear in the level 0 Dirac coordinate representations of the hyperbolic Cayley-Dickson algebras form

TABLE 3.3. The standard basis elements of $\mathbb{G}$, the ring of hyperbolic octonions, written in Hamilton coordinates.

Symbol	Hamilton Coordinates		
Level 3	Level 2	Level 1	Level 0
1	$(1,0)$	$((1,0),(0,0))$	$(((1,0),(0,0)),((0,0),(0,0)))$
$\mathbb{i}$	$(\mathbb{i},0)$	$((\mathsf{j},0),(0,0))$	$(((0,1),(0,0)),((0,0),(0,0)))$
$\mathbb{j}$	$(\mathbb{j},0)$	$((0,1),(0,0))$	$(((0,0),(1,0)),((0,0),(0,0)))$
$\mathbb{k}$	$(\mathbb{k},0)$	$((0,\mathsf{j}),(0,0))$	$(((0,0),(0,1)),((0,0),(0,0)))$
$\mathbb{l}$	$(0,1)$	$((0,0),(1,0))$	$(((0,0),(0,0)),((1,0),(0,0)))$
$\mathbb{i}\mathbb{l}$	$(0,\mathbb{i})$	$((0,0),(\mathsf{j},0))$	$(((0,0),(0,0)),((0,1),(0,0)))$
$\mathbb{j}\mathbb{l}$	$(0,\mathbb{j})$	$((0,0),(0,1))$	$(((0,0),(0,0)),((0,0),(1,0)))$
$\mathbb{k}\mathbb{l}$	$(0,\mathbb{k})$	$((0,0),(0,\mathsf{j}))$	$(((0,0),(0,0)),((0,0),(0,1)))$

TABLE 3.4. The standard basis of $\mathbb{G}$, the ring of hyperbolic octonions, written in Dirac coordinates.

Symbol	Dirac Coordinates		
Level 3	Level 2	Level 1	Level 0
1	$[1,\ 1]$	$[[1,\ 1],[\ 1,\ 1]]$	$[[[+,+],[+,+]],[[+,+],[+,+]]]$
$\mathbb{i}$	$[\mathbb{i},\ \mathbb{i}]$	$[[\mathsf{j},\ \mathsf{j}],[\ \mathsf{j},\ \mathsf{j}]]$	$[[[+,-],[+,-]],[[+,-],[+,-]]]$
$\mathbb{j}$	$[\mathbb{j},\ \mathbb{j}]$	$[[1,-1],[\ 1,-1]]$	$[[[+,+],[-,-]],[[+,+],[-,-]]]$
$\mathbb{k}$	$[\mathbb{k},\ \mathbb{k}]$	$[[\mathsf{j},-\mathsf{j}],[\ \mathsf{j},-\mathsf{j}]]$	$[[[+,-],[-,+]],[[+,-],[-,+]]]$
$\mathbb{l}$	$[1,-1]$	$[[1,\ 1],[-1,-1]]$	$[[[+,+],[+,+]],[[-,-],[-,-]]]$
$\mathbb{i}\mathbb{l}$	$[\mathbb{i},-\mathbb{i}]$	$[[\mathsf{j},\ \mathsf{j}],[-\mathsf{j},-\mathsf{j}]]$	$[[[+,-],[+,-]],[[-,+],[-,+]]]$
$\mathbb{j}\mathbb{l}$	$[\mathbb{j},-\mathbb{j}]$	$[[1,-1],[-1,\ 1]]$	$[[[+,+],[-,-]],[[-,-],[+,+]]]$
$\mathbb{k}\mathbb{l}$	$[\mathbb{k},-\mathbb{k}]$	$[[\mathsf{j},-\mathsf{j}],[-\mathsf{j},\ \mathsf{j}]]$	$[[[+,-],[-,+]],[[-,+],[+,-]]]$

the set of *Walsh functions*. In fact, the natural order used in the above tables to list the basis elements of the hyperbolic CD algebras corresponds to the *natural* or *Hadamard* order of the Walsh functions. This interpretation of the Cayley-Dickson algebra basis elements as Walsh functions should lead to another way to calculate the products of the Cayley-Dickson algebras similar to the results obtained by Hagmark and Lounesto in References [13] and [24, pp. 279 ff.].

4. A Matrix Representation for the Hyperbolic Quaternion Algebra

Let us try to define a matrix representation of the hyperbolic quaternion algebra $\mathbb{M}$ which is isomorphic to the orientation congruent algebra $\mathcal{OC}_2$. We need four matrices that might correspond to the four basis elements of $\mathbb{M}$, 1, $\mathbb{i} \cong \mathbf{e}_1$, $\mathbb{j} \cong \mathbf{e}_2$, and $\mathbb{k} \cong \mathbf{e}_{12}$.

Perhaps the matrices

$$\mathbf{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A} := \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \quad \mathbf{B} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{C} := \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \tag{4.1}$$

might represent the basis elements of $\mathbb{M}$ with the correspondences $\mathbf{I} \leftrightarrow 1$, $\mathbf{A} \leftrightarrow \mathbb{i}$, $\mathbf{B} \leftrightarrow \mathbb{j}$, $\mathbf{C} \leftrightarrow \mathbb{k}$. But under what product?

Let the set of square matrices and their matrix algebras over $\mathbb{R}$, $\mathbb{C}$, and Hamilton's quaternions $\mathbb{H}$, be written as $\mathrm{Mat}(\mathbb{R}, n)$, $\mathrm{Mat}(\mathbb{C}, n)$, and $\mathrm{Mat}(\mathbb{H}, n)$, respectively. It is among these algebras that faithful matrix representations of the real Clifford algebras $\mathcal{C}\ell_{p,q}$ are found. All elements of these matrix algebras *do* associate because their multiplication is the usual matrix product and each entry in these matrices is a member of an associative algebra, but the elements of $\mathbb{M}$ *do not* associate under the orientation congruent product. Therefore, the usual matrix algebras and their products cannot represent the algebra $\mathbb{M}$ and its product.

Before attempting to define an appropriate product, we review some standard definitions and notation.

For any matrix $\mathbf{P}$ with entries from the complex field $\mathbb{C}$, let an overline as in $\overline{\mathbf{P}}$ mean the (matrix) *complex conjugation* of $\mathbf{P}$ defined as the complex conjugation of every element of $\mathbf{P}$. Also, let a lower case superscript t as in $\mathbf{P}^{\mathrm{t}}$ mean the *transposition* of $\mathbf{P}$. Finally, let an upper case superscript H as in $\mathbf{P}^{\mathrm{H}}$ mean the *Hermitian conjugation* of $\mathbf{P}$ defined as the transpose of the matrix conjugate, or, equivalently, the conjugate of the matrix transpose

$$\mathbf{P}^{\mathrm{H}} := (\overline{\mathbf{P}})^{\mathrm{t}} = \overline{\mathbf{P}^{\mathrm{t}}}. \tag{4.2}$$

We now define the (left) *Hermitian conjugate product,* denoted by a circled star $\circledast$, so that for all conforming matrices $\mathbf{P}$ and $\mathbf{Q}$

$$\mathbf{P} \circledast \mathbf{Q} := \mathbf{P}^{\mathrm{H}}\mathbf{Q}. \tag{4.3}$$

Here, as usual, juxtaposition indicates the standard associative matrix product.

The reader may verify that the Hermitian conjugate product of any two same or different matrices from the set $\{\,\mathbf{A}, \mathbf{B}, \mathbf{C}\,\} \subseteq \mathrm{Mat}(\mathbb{C}, 2)$ corresponds to the product of the corresponding elements of $\mathbb{M}$. Unfortunately, this correspondence of products breaks down when one of the matrices from this set is the Hermitian conjugate multiplier of the identity matrix $\mathbf{I}$ as in $\mathbf{A} \circledast \mathbf{I} = \mathbf{A}^{\mathrm{H}} \neq \mathbf{A}$. However, a fix is possible. It is based on one of the extended Kauffman product duality formulas for elliptic and hyperbolic quaternions which was proved in Table 2.2.

Motivated by the partial success of $\mathbf{I}$, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ as matrix representations for $\mathbb{M}$ let us now try translating expressions in Dirac coordinates into 2×2 matrices of the same form as $\mathbf{I}$, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ according to the scheme

$$[w, x] \mapsto \frac{1}{2} \begin{pmatrix} (\overline{w} + x) & (-w + \overline{x}) \\ (\overline{w} - x) & (w + \overline{x}) \end{pmatrix}. \tag{4.4}$$

As it was designed to do this transformation produces the following correspondence between the standard 1, $\mathbb{i}$, $\mathbb{j}$, and $\mathbb{k}$ basis elements of the hyperbolic quaternions in Dirac coordinates and double number 2×2 matrices

$$
\begin{aligned}
1 &\cong [1,1] \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \mathbb{i} &\cong [\mathsf{j},\mathsf{j}] \mapsto \begin{pmatrix} 0 & -\mathsf{j} \\ -\mathsf{j} & 0 \end{pmatrix}, \\
\mathbb{j} &\cong [1,-1] \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and } & \mathbb{k} &\cong [\mathsf{j},-\mathsf{j}] \mapsto \begin{pmatrix} -\mathsf{j} & 0 \\ 0 & \mathsf{j} \end{pmatrix}.
\end{aligned} \tag{4.5}
$$

As just stated the imaginary numbers in these matrices are double numbers from the ring $\mathbb{D}$. However, if instead they were imaginary numbers from the complex field $\mathbb{C}$ we would have exactly the four matrices $\mathbf{I}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ defined in Equation (4.1). Therefore, we define the matrix representations of the hyperbolic quaternions $\mathbb{M}$ to be just so. We are encouraged to make this move not only by the partial success of $\mathbf{I}$, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ as matrix representations for $\mathbb{M}$, but also by the extended Kauffman product duality formulas of Equations (2.2) defining hyperbolic quaternion multiplication in terms of elliptic quaternion multiplication and vice versa.

We now have the following correspondence

$$
\begin{aligned}
1 &\cong [1,1] \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \mathbb{i} &\cong [\mathsf{j},\mathsf{j}] \mapsto \begin{pmatrix} 0 & -\mathsf{i} \\ -\mathsf{i} & 0 \end{pmatrix}, \\
\mathbb{j} &\cong [1,-1] \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and } & \mathbb{k} &\cong [\mathsf{j},-\mathsf{j}] \mapsto \begin{pmatrix} -\mathsf{i} & 0 \\ 0 & \mathsf{i} \end{pmatrix}.
\end{aligned} \tag{4.6}
$$

This correspondence defines a linear transformation which we call m, and we call matrices of this form m-matrices. Under the m transformation we may write, for example,

$$
m(\mathbb{i}) = m([\mathsf{j},\mathsf{j}]) = \begin{pmatrix} 0 & -\mathsf{i} \\ -\mathsf{i} & 0 \end{pmatrix}. \tag{4.7}
$$

The linear transformation m establishes at least a vector space isomorphism from the hyperbolic quaternions, or their Dirac coordinates, to the space of complex 2×2 matrices $\mathrm{Mat}(\mathbb{C}, 2)$.

Here we digress to discuss some properties of m-matrices and their conversion to and from Dirac coordinates. These matrices obey the following equation which explicitly shows that they have only two degrees of freedom

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}. \tag{4.8}
$$

It is easy to see that all independent pairs of entries are those in which both entries are not on the same main or minor diagonal.

The next diagram illustrates a "Chinese restaurant" scheme for translating m-matrices back into Dirac coordinates: take one from the first column and one from the second column

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left[\begin{array}{c|c} \overline{a} - b & a + \overline{b} \\ \overline{a} + \overline{c} & a - c \\ d - b & \overline{d} + \overline{b} \\ d + \overline{c} & d - \overline{c} \end{array}\right]. \tag{4.9}
$$

Thus, for matrices of this form, a few of the 16 equivalent corresponding Dirac coordinate expressions are given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [d-b, a-c] = [d+\overline{c}, a+\overline{b}] = [\overline{a}-b, a+\overline{b}] = [d-b, \overline{d}+\overline{b}]. \tag{4.10}$$

It turns out that the m-matrices for $\mathbb{i}$, $\mathbb{j}$, and $\mathbb{k}$ are related to the Pauli spin matrices, $\sigma_1, \sigma_2, \sigma_3$ which form the standard matrix representations of the elliptic quaternions i, j, and k as isomorphic to the bivectors -$\mathbf{e}_{23}$, -$\mathbf{e}_{31}$, and -$\mathbf{e}_{12}$, respectively, of $\mathcal{C}\ell_3$ [24, pp. 54-56]

$$\begin{aligned} m(\mathbb{i}) &= -\mathrm{i}\sigma_1 \cong i \cong -\mathbf{e}_{23} \in \mathcal{C}\ell_3, \\ m(\mathbb{j}) &= -\mathrm{i}\sigma_2 \cong j \cong -\mathbf{e}_{31} \in \mathcal{C}\ell_3, \\ m(\mathbb{i}) &= -\mathrm{i}\sigma_3 \cong k \cong -\mathbf{e}_{12} \in \mathcal{C}\ell_3. \end{aligned} \tag{4.11}$$

The Hermitian conjugate of the m-matrices is given explicitly by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\mathrm{H}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{4.12}$$

Lounesto [24, pp. 54-56] informs us that the Hermitian conjugation of these matrices is equivalent to the reversion, or, since the m-matrices represent bivectors, the Clifford conjugation, of their corresponding Clifford algebra elements in $\mathcal{C}\ell_3$. It also easy to see that Hermitian conjugation of the m-matrices is equivalent to quaternion conjugation of the corresponding elements of the quaternion algebra. We note in passing that Hermitian conjugation of these matrices is an example of a *simplectic involution* defining a split composition algebra over $\mathrm{Mat}(\mathbb{C}, 2)$ as described by E. N. Kuz′min and Ivan P. Shestakov [21, p. 218].

Let p and q be hyperbolic quaternions with m-matrix representations $m(p)$ and $m(p)$, respectively. Also, redefine the circled star $\circledast$ used previously to represent the Hermitian conjugate product to now represent the nonassociative matrix product corresponding to the hyperbolic quaternion product Then the product of the hyperbolic quaternions p and q may be expressed as

(4.13)
$$m(p) \circledast m(q) = \frac{1}{2}\left(m(q)\, m(p) + m(q)\, m(p)^{\mathrm{H}} + m(q)^{\mathrm{H}}\, m(p) - m(q)^{\mathrm{H}}\, m(p)^{\mathrm{H}}\right).$$

Proof. The right hand side of the above equation is just one of the extended Kauffman product duality formulas from Equations (2.2) expressing the hyperbolic quaternion product in terms of the elliptic quaternion product. Here, however, it has been translated into the standard matrices representing the even subalgebra of $\mathcal{C}\ell_3$ which in turn is isomorphic to the elliptic quaternion algebra. Also, as noted above, in the standard matrix representation for $\mathcal{C}\ell_3$, Hermitian conjugation is the matrix operation corresponding to quaternion conjugation. The result follows. □

Some example calculations follow.

$$\begin{aligned}
\mathbb{ij} \mapsto m(\mathbb{i}) \circledast m(\mathbb{j}) &= \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \circledast \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
m(\mathbb{i}) \circledast m(\mathbb{j}) &= \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}^{\mathrm{H}} \\
&\quad + \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\mathrm{H}}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\mathrm{H}}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}^{\mathrm{H}} \\
&= \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \\
&\quad + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \\
&= \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \mapsto \mathbb{k}.
\end{aligned} \tag{4.14}$$

(4.15)

$$\begin{aligned}
\mathbb{i}\,\mathbb{i} \mapsto m(\mathbb{i}) \circledast m(\mathbb{i}) &= \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \circledast \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \\
m(\mathbb{i}) \circledast m(\mathbb{i}) &= \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}^{\mathrm{H}} \\
&\quad + \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}^{\mathrm{H}}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}^{\mathrm{H}}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}^{\mathrm{H}} \\
&= \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \\
&\quad + \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto 1.
\end{aligned}$$

5. The Hyperbolic Cayley-Dickson Algebras and Generalizations

We are now ready to carry out the final steps in our program to develop, using duality, a recursive definition of the generalized hyperbolic Cayley-Dickson algebras from the standard recursive definition of the elliptic Cayley-Dickson algebras. However, we can go beyond the elliptic case by defining the *ultrageneralized* hyperbolic Cayley-Dickson algebras in the last part of this section. In fact, we are forced to go beyond the elliptic case because certain types of split ultrageneralized hyperbolic quaternion algebras, for example, are needed as factors in the representation of the orientation congruent algebras $\mathcal{OC}_{p,q}$ as tensor products of algebras in Section 5.

We have already carried out the preliminary step in Section 2 by using the first extended Kauffman product duality formula, Equation 2.2a, to determine the hyperbolic quaternion m-matrix representation and its nonassociative matrix product. Here we use these complex matrices to find a recursive hyperbolic-elliptic quaternion product formula which has hyperbolic quaternion multiplication on the left hand side and complex number multiplication on the right. When the matrix representation is converted back to Dirac coordinates, and then to Hamilton coordinates, we obtain the recursive *Cayley-Dickson algebra product duality formula* which is Equation (5.3a) of Theorem 5.1. By an argument similar to that in the proof of Theorem 2.3, this formula is valid at any Cayley-Dickson algebra level, not just the quaternion one. Theorem 5.1 expressing this formula and its dual is the first major result of this section.

Then, we use the second extended Kauffman product duality formula, Equation 2.2b, to expand each complex number product occurring in this formula into a double number internal product. This result is also valid at any level beyond the quaternions by an argument similar to that in the proof of Theorem 2.3. This, the second major result of this section, is given in Theorem 5.2 as the *generalized hyperbolic Cayley-Dickson product algebra formula*—an expression recursively defining the hyperbolic Cayley-Dickson algebras as pairs of numbers from the hyperbolic Cayley-Dickson algebra at the next lower level.

Finally, we obtain our last major result of this section by modifying the recursive formula for the hyperbolic Cayley-Dickson algebra product to define the *ultrageneralized hyperbolic Cayley-Dickson algebras* with the recursive product formula given in Definition 5.4. As mentioned above, some algebras of this type are required factors in the tensor product of algebras representation of the orientation congruent algebras discussed in Section 6.

Recall the following result of an m transformation on the hyperbolic quaternion $q = [w, x]$ in Dirac coordinates

$$m([w, x]) = \frac{1}{2}\begin{pmatrix} (\overline{w} + x) & (-w + \overline{x}) \\ (\overline{w} - x) & (w + \overline{x}) \end{pmatrix}. \tag{5.1}$$

The computation of the product $q_3 = q_1 q_2$ of the general hyperbolic quaternions $q_1 = [w_1, x_1]$ and $q_2 = [w_2, x_2]$ in their m-matrix representations begins as follows

$$
\begin{aligned}
m(q_3) &= m(q_1) \circledast m(q_2)\\
&= \frac{1}{2}\begin{pmatrix}(\overline{w}_1 + x_1) & (-w_1 + \overline{x}_1)\\ (\overline{w}_1 - x_1) & (w_1 + \overline{x}_1)\end{pmatrix} \circledast \frac{1}{2}\begin{pmatrix}(\overline{w}_2 + x_2) & (-w_2 + \overline{x}_2)\\ (\overline{w}_2 - x_2) & (w_2 + \overline{x}_2)\end{pmatrix}\\
&= \frac{1}{4}\begin{pmatrix}(\overline{w}_2 + x_2) & (-w_2 + \overline{x}_2)\\ (\overline{w}_2 - x_2) & (w_2 + \overline{x}_2)\end{pmatrix}\begin{pmatrix}(\overline{w}_1 + x_1) & (-w_1 + \overline{x}_1)\\ (\overline{w}_1 - x_1) & (w_1 + \overline{x}_1)\end{pmatrix}\\
&\qquad + \frac{1}{4}\begin{pmatrix}(\overline{w}_2 + x_2) & (-w_2 + \overline{x}_2)\\ (\overline{w}_2 - x_2) & (w_2 + \overline{x}_2)\end{pmatrix}\begin{pmatrix}(\overline{w}_1 + x_1) & (-w_1 + \overline{x}_1)\\ (\overline{w}_1 - x_1) & (w_1 + \overline{x}_1)\end{pmatrix}^{\mathrm{H}}\\
&\qquad + \frac{1}{4}\begin{pmatrix}(\overline{w}_2 + x_2) & (-w_2 + \overline{x}_2)\\ (\overline{w}_2 - x_2) & (w_2 + \overline{x}_2)\end{pmatrix}^{\mathrm{H}}\begin{pmatrix}(\overline{w}_1 + x_1) & (-w_1 + \overline{x}_1)\\ (\overline{w}_1 - x_1) & (w_1 + \overline{x}_1)\end{pmatrix}\\
&\qquad - \frac{1}{4}\begin{pmatrix}(\overline{w}_2 + x_2) & (-w_2 + \overline{x}_2)\\ (\overline{w}_2 - x_2) & (w_2 + \overline{x}_2)\end{pmatrix}^{\mathrm{H}}\begin{pmatrix}(\overline{w}_1 + x_1) & (-w_1 + \overline{x}_1)\\ (\overline{w}_1 - x_1) & (w_1 + \overline{x}_1)\end{pmatrix}^{\mathrm{H}}
\end{aligned}
\tag{5.2}
$$

The rest of this calculation is extremely tedious and prone to accumulate many errors if done by hand. So we let Mathematica finish the work. The result is expressed in Hamilton coordinates with an implicit product on the right hand side which is that of the complex numbers, or more generally, that of the elliptic Cayley-Dickson algebra one level lower than the hyperbolic one on the left hand side. This equation, which is one of the *Cayley-Dickson algebra product duality formulas*, is, by duality, also valid with an elliptic left hand side product and a hyperbolic right hand side product.

At this first stop of this leg of our journey we have penetrated deeply into the land of Ogs. We find that, in this land beyond the vast realm of the Wizard Kauffman of Chicago, any number of nested snakes may bite their tails. The first major result of this section is

Theorem 5.1 (The Cayley-Dickson Algebra Product Duality Formulas)**.** *The products on the right-hand-sides of the following equations are not shown explicitly; however, they are dual to those on the left-hand-sides.*

$$
\begin{aligned}
(w_1, x_1) \circledast (w_2, x_2) = \frac{1}{2}\big(&w_2 w_1 + \overline{w}_2 w_1 + w_2 \overline{w}_1 - \overline{w}_2 \overline{w}_1 + 2 x_2 \overline{x}_1,\\
&2(x_2 w_1 + \overline{w}_2 x_1)\big)
\end{aligned}
\tag{5.3a}
$$

$$
\begin{aligned}
(w_1, x_1) * (w_2, x_2) = \frac{1}{2}\big(&w_2 w_1 + \overline{w}_2 w_1 + w_2 \overline{w}_1 - \overline{w}_2 \overline{w}_1 + 2 x_2 \overline{x}_1,\\
&2(x_2 w_1 + \overline{w}_2 x_1)\big).
\end{aligned}
\tag{5.3b}
$$

Proof. Most of the proof of Equation (5.3a) was given in the text immediately before this theorem. Furthermore, by an argument similar to that in the proof of Theorem 2.3, once this formula is verified for elliptic or hyperbolic octonions at level 3 on the left hand side, it is valid at any Cayley-Dickson algebra level. This verification is left to the reader. Equation (5.3b) follows by the duality of Theorem 2.3. □

Note the formal equivalence of Equations (5.3a) and (5.3b) in Theorem 5.1. By combining pairs of mutually conjugate second factors in each term in Equations

(5.3) we obtain the following versions of the Cayley-Dickson product duality formulas

$$
\begin{aligned}
(w_1, x_1) \circledast (w_2, x_2) = \frac{1}{2}\big(&w_2(w_1 + \overline{w}_1) + \overline{w}_2(w_1 - \overline{w}_1) + 2x_2\overline{x}_1, \\
&2(x_2 w_1 + \overline{w}_2 x_1)\big).
\end{aligned} \tag{5.4a}
$$

$$
\begin{aligned}
(w_1, x_1) * (w_2, x_2) = \frac{1}{2}\big(&w_2(w_1 + \overline{w}_1) + \overline{w}_2(w_1 - \overline{w}_1) + 2x_2\overline{x}_1, \\
&2(x_2 w_1 + \overline{w}_2 x_1)\big).
\end{aligned} \tag{5.4b}
$$

Here are some example calculations for hyperbolic quaternions

$$
\begin{aligned}
\mathbb{ij} &\mapsto (\mathsf{i}, 0) \circledast (0, 1) = \frac{1}{2}(0(\mathsf{i} - \mathsf{i}) + 0(\mathsf{i} + \mathsf{i}) + 2(1 \cdot 0), 2(1\mathsf{i} + 0 \cdot 0) = (0, \mathsf{i}) \mapsto \mathbb{k}, \\
\mathbb{ji} &\mapsto (0, 1) \circledast (\mathsf{i}, 0) = \frac{1}{2}(\mathsf{i}(0 + 0) - \mathsf{i}(0 - 0) + 2(0 \cdot 1), 2(0 \cdot 0 - \mathsf{i} \cdot 1) = (0, -\mathsf{i}) \mapsto -\mathbb{k}, \\
\mathbb{ii} &\mapsto (\mathsf{i}, 0) \circledast (\mathsf{i}, 0) = \frac{1}{2}(\mathsf{i}(\mathsf{i} - \mathsf{i}) - \mathsf{i}(\mathsf{i} + \mathsf{i}) + 2(0 \cdot 0), 2(0\mathsf{i} - \mathsf{i}0) = (1, 0) \mapsto 1.
\end{aligned}
$$

This formula should also work with hyperbolic octonions. Recalling that the implicit product on the right hand side of the following equations is elliptic quaternion multiplication, here are some example calculations for hyperbolic octonions

$$
\begin{aligned}
\mathbb{ij} &\mapsto (i, 0) \circledast (j, 0) = \frac{1}{2}(j(i - i) - j(i + i) + 2(0 \cdot 0), 2(0i - j0) = (k, 0) \mapsto \mathbb{k}, \\
\mathbb{ji} &\mapsto (j, 0) \circledast (i, 0) = \frac{1}{2}(i(j - j) - i(j + j) + 2(0 \cdot 0), 2(0j - i0) = (-k, 0) \mapsto -\mathbb{k}, \\
\mathbb{ii} &\mapsto (i, 0) \circledast (i, 0) = \frac{1}{2}(i(i - i) - i(i + i) + 2(0 \cdot 0), 2(0i - i0) = (1, 0) \mapsto 1, \\
\mathbb{il} &\mapsto (i, 0) \circledast (0, 1) = \frac{1}{2}(0(i - i) - 0(i + i) + 2(1 \cdot 0), 2(1i + 0 \cdot 0) = (0, i) \mapsto \mathbb{il}, \\
\mathbb{li} &\mapsto (0, 1) \circledast (i, 0) = \frac{1}{2}(i(0 + 0) - i(0 - 0) + 2(0 \cdot 1), 2(0 \cdot 0 - i1) = (0, -i) \mapsto -\mathbb{il}, \\
(\mathbb{il})\mathbb{i} &\mapsto (0, i) \circledast (i, 0) = \frac{1}{2}(i(0 + 0) - i(0 - 0) + 2(0(-i)), 2(0 \cdot -ii) = (0, 1) \mapsto \mathbb{l}.
\end{aligned}
$$

Now we are ready to obtain the second major result of this section, the *generalized hyperbolic Cayley-Dickson algebra product formula*, in which all products are hyperbolic ones, including the implicit products on the right hand side. At this, our second stop in this leg of our journey into the land of Ogs, we find that the nested snakes have learned to keep their heads intact while miraculously turning the rest of their bodies inside out somewhere between their heads and tails *and* still managing to bite those inside-out tails.

We start the derivation that proves this theorem by applying the extended Kauffman product duality formula of Equation (2.2b) to the right hand side of Cayley-Dickson algebra product duality formula of Equation (5.3). Then, we simplify in Mathematica. Finally, after adding the factor γ (which, for example, is chosen as $+1$ (resp., -1) to define the ungeneralized (resp., split) hyperbolic quaternions in terms of pairs of double numbers) we obtain the *generalized hyperbolic Cayley-Dickson product formula*

Theorem 5.2 (The Generalized Hyperbolic Cayley-Dickson Algebra Product Formula)**.**

$$
\begin{aligned}
(w_1, x_1) \circledast (w_2, x_2) = \frac{1}{2}\big(2w_1w_2 + \gamma(x_1x_2 + \overline{x}_1x_2 - x_1\overline{x}_2 + \overline{x}_1\overline{x}_2),\\
(w_1x_2 + \overline{w}_1x_2 + w_1\overline{x}_2 - \overline{w}_1\overline{x}_2)\\
+ (x_1w_2 - \overline{x}_1w_2 + x_1\overline{w}_2 + \overline{x}_1\overline{w}_2)\big)
\end{aligned}
\tag{5.5a}
$$

which can be regrouped as

$$
\begin{aligned}
= \frac{1}{2}\Big(2w_1w_2 + \gamma\big(x_1(x_2 - \overline{x}_2) + \overline{x}_1(x_2 + \overline{x}_2)\big),\\
w_1(x_2 + \overline{x}_2) + \overline{w}_1(x_2 - \overline{x}_2)\\
+ (x_1 - \overline{x}_1)w_2 + (x_1 + \overline{x}_1)\overline{w}_2)\Big).
\end{aligned}
\tag{5.5b}
$$

Proof. Again, most of the proof was given in the text immediately before this theorem. It only remains to note that, once more, by an argument similar to that in the proof of Theorem 2.3, this formula is valid at any Cayley-Dickson algebra level, once it has been verified for elliptic or hyperbolic octonions at level 3 on the left hand side. This verification is again left to the reader. □

To express the Clifford algebras $\mathcal{C}\ell_{p,q}$ as tensor products of algebras we need only four factor algebras, the complex and dual number algebras, $\mathbb{C}$ and $\mathbb{D}$, as well as the ungeneralized and split elliptic quaternions, $\mathbb{H}$ and $\underset{\sim}{\mathbb{H}} \cong \mathrm{Mat}(\mathbb{R}, 2)$. For details see, for example, [24, Ch. 16], [33], [34], [35, Ch. 15], or [36]. However, in the hyperbolic case, in order to factor the orientation congruent algebras $\mathcal{OC}_{p,q}$, we need even further generalized (*ultrageneralized*) types of quaternions than just the ungeneralized and split versions that were sufficient in the elliptic case.

These ultrageneralized hyperbolic quaternion algebras are dependent on three more parameters than the one, namely γ, whose value determines whether the generalized elliptic and hyperbolic quaternions reduce to their ungeneralized version or not. In the following it is convenient to first rename γ as α. These parameters, four in all, α, β, γ, and δ, may, most generally, take the value of any real number in a similar way to that discussed in Shafarevich's book [47, pp. 93 f., 201] for the generalized (elliptic) quaternion algebras and, in his words, the "generalised Cayley algebras" (which are our generalized elliptic octonion algebras). Thus, the algebras of the generalized as well as the ultrageneralized hyperbolic quaternions are determined by α, β, γ, and δ. The resulting multiplication table for the ultrageneralized hyperbolic quaternions is displayed in Table 5.1.

The distribution of these parameters among the products of this multiplication table is not arbitrary, but is correlated with the role of the four parameters in the defining Equation (5.7) for the ultrageneralized hyperbolic Cayley-Dickson algebras. Equation (5.7) is presented soon below.

In fact, the above presentation of the ultrageneralized hyperbolic quaternions by their multiplication table, although easier to assimilate, has the cart before the horse. This is because it is the following two requirements that dictate the distribution of the additional parameters β, γ, and δ (beyond α, which is the original γ for the generalized algebras) in Table 5.1: (1) the additional parameters must be naturally incorporated into Equation (5.7) defining the ultrageneralized hyperbolic CD algebras; and (2) the resulting ultrageneralized hyperbolic CD algebras must

Table 5.1. The multiplication table for the ultrageneralized hyperbolic quaternions. It is dependent on the values from $\mathbb{R}$ assigned to the parameters α, β, δ, and γ. E.g., the all-positive choice $\alpha = \beta = \gamma = \delta = +1$ gives the algebra $\mathbb{M}$ of Macfarlane's (ungeneralized) hyperbolic quaternions; while the single-negative choice $\alpha = -1$ gives the algebra $\not{\mathbb{M}}$ of split hyperbolic quaternions. The product is symbolized as $\circledast$, circled to indicate a hyperbolic algebra.

		b			
	$a \circledast b$	1	$\mathbb{i}$	$\mathbb{j}$	$\mathbb{k}$
a	1	1	$\mathbb{i}$	$\mathbb{j}$	$\mathbb{k}$
	$\mathbb{i}$	$\mathbb{i}$	1	$\gamma\delta\,\mathbb{k}$	$-\gamma\mathbb{j}$
	$\mathbb{j}$	$\mathbb{j}$	$-\gamma\delta\,\mathbb{k}$	α	$\alpha\beta\,\mathbb{i}$
	$\mathbb{k}$	$\mathbb{k}$	$\gamma\mathbb{j}$	$-\alpha\beta\,\mathbb{i}$	α

include some quaternion algebras that are suitable factors in Section 6's representations of the orientation congruent algebras as tensor products of algebras.

We have been developing Equation (5.7) for the recursive product formula of the ultrageneralized hyperbolic CD algebras in terms of the ultrageneralized hyperbolic quaternions. However, it is important to note that this equation defines a whole sequence of ultrageneralized hyperbolic Cayley-Dickson algebras, not just the ultrageneralized hyperbolic quaternion ones of Table 5.1.

We are about to incorporate the parameters α, β, γ, and δ into the recursive product formula for the ultrageneralized hyperbolic Cayley-Dickson algebras to effect the mutations of products shown in Table 5.1. But first, we must define the *imaginary part multiplication operator* $C(\bullet)$.

Definition 5.3 (Imaginary Part Multiplication Operator)**.** The *imaginary part multiplication operator* $C(\bullet)$ acting on w is written as a superscript as in $w^{C(\bullet)}$. For some parameter, say $\gamma \in \mathbb{R}$, it operates on the element w of the next lower (elliptic or hyperbolic) Cayley-Dickson algebra by multiplying the imaginary part of w by γ as in

$$w^{C(\gamma)} := \frac{1}{2}\Big(\big(w + \overline{w}\big) + \gamma\big(w - \overline{w}\big)\Big) = \operatorname{Re} w + \gamma \operatorname{Im} w. \tag{5.6}$$

We have now arrived at the third and last stop in this leg of our tour of the land of Ogs. Here we find that the nested snakes have learned the trick of cutting out various sections of their heads in intricate patterns and rejoining them to the rest of their heads upside down as they bite their inside-out tails. In this, the final major result of this section, recall that the original parameter γ of Theorem 5.2, and which determines the ungeneralized or generalized character of a hyperbolic CD algebra, remains as the simple multiplier it previously was, but it is now renamed to α. We define the *ultrageneralized hyperbolic Cayley-Dickson algebras* with the following recursive product formula.

Definition 5.4 (The Ultrageneralized Hyperbolic Cayley-Dickson Algebra Product Formula)**.**

$$\begin{aligned}(w_1,x_1)\circledast(w_2,x_2)=\frac{1}{2}\Big(&2w_1w_2+\alpha\big(x_1(x_2-\overline{x}_2)+\overline{x}_1(x_2+\overline{x}_2)\big)^{C(\beta)},\\ &\Big(w_1^{C(\gamma)}(x_2+\overline{x}_2)+\overline{w}_1^{C(\gamma)}(x_2-\overline{x}_2)\\ &+(x_1-\overline{x}_1)w_2^{C(\gamma)}+(x_1+\overline{x}_1)\overline{w}_2^{C(\gamma)}\Big)^{C(\delta)}\Big).\end{aligned}\tag{5.7}$$

We again emphasize that the recursive product formula in Equation (5.7) defines a whole sequence of ultrageneralized hyperbolic CD algebras, not just the ultrageneralized hyperbolic quaternion ones. However, for the ultrageneralized hyperbolic Cayley-Dickson algebras over the real field $\mathbb{R}$, Equation (5.7) reduces at level 1 to

$$(w_1,x_1)\circledast(w_2,x_2)=(w_1w_2+\alpha x_1x_2, w_1x_2+x_1w_2).$$

It is extremely convenient that at level 1 only the parameter α remains in this equation. The values of β, γ, and δ are irrelevant since conjugation of a real number has no effect. Thus we avoid the possible production of equations which contradict the definition of 1 as the multiplicative identity. For the ungeneralized double number ring $\mathbb{D}$ (identical to the split complex ring $\mathbb{C}\!\!\!\backslash$) the forbidden equations are $1\mathsf{j}=\mathsf{j}1=-\mathsf{j}$; while for split double number field $\mathbb{D}\!\!\!\backslash$ (identical to the complex field $\mathbb{C}$) the forbidden equations are $1\mathsf{i}=\mathsf{i}1=-\mathsf{i}$.

Equation (5.8a) of the following lemma is useful for calculations with computer-assisted algebra systems that come with standard packages for the elliptic quaternions, such as Mathematica. This equation allows the definition of the ultrageneralized hyperbolic quaternion operations in terms of the elliptic quaternion ones. Thus, the tediousness of verifying the results of the next section may be reduced by assigning most of the calculations to the computer.

Lemma 5.5 (The Ultrageneralized Cayley-Dickson Algebra Product Duality Formulas)**.** *Consistent with our earlier presentation in Theorem 5.1, the products on the right-hand-sides of the following equations are not shown explicitly; however, they are dual to those on the left-hand-sides.*

$$\begin{aligned}(w_1,x_1)\circledast(w_2,x_2)=\frac{1}{2}\Big(&w_2w_1+\overline{w}_2w_1+w_2\overline{w}_1-\overline{w}_2\overline{w}_1+2\alpha(x_2\overline{x}_1)^{C(\beta)},\\ &2\Big(x_2w_1^{C(\gamma)}+\overline{w}_2^{C(\gamma)}x_1\Big)^{C(\delta)}\Big)\end{aligned}\tag{5.8a}$$

$$\begin{aligned}(w_1,x_1)*(w_2,x_2)=\frac{1}{2}\Big(&w_2w_1+\overline{w}_2w_1+w_2\overline{w}_1-\overline{w}_2\overline{w}_1-2\alpha(x_2\overline{x}_1)^{C-(\beta)},\\ &2\Big(x_2w_1^{C(-\gamma)}+\overline{w}_2^{C(-\gamma)}x_1\Big)^{C(-\delta)}\Big)\end{aligned}\tag{5.8b}$$

Proof. The proof is by direct verification of the equivalence of Equation (5.7) of Definition 5.4 with Equation (5.8a) above for the ultrageneralized hyperbolic quaternions, followed by its dualization into Equation (5.8b). The validity of Equation (5.8a) for the ultrageneralized hyperbolic n-level Cayley-Dickson algebras for any n, not just for $n=2$ and the ultrageneralized hyperbolic quaternions, follows again by an argument similar to that used in the proof of Theorem 2.3. $\square$

Note that the formal equivalence of Equations (5.8a) and (5.8b) in Lemma 5.5 is spoiled by the sign changes required to maintain dual sign conventions for the parameters β, γ, and δ that are consistent with the dual sign convention of α. The dual sign convention for α is determined by Definition 1.2 in which its parameter γ is equivalent to α.

While the parameters α, β, γ, and δ that appear as arguments of the imaginary part multiplication operator $C(\bullet)$ in Equation (5.4) defining the ultrageneralized hyperbolic Cayley-Dickson algebras may take any values in $\mathbb{R}$, when these four parameters are restricted to the two values -1 and $+1$, we obtain the *standard, nondegenerate, ultrageneralized hyperbolic Cayley-Dickson algebras*. If, in addition, the level of hyperbolic CD algebras is restricted to the value 2, we obtain *standard, nondegenerate, ultrageneralized hyperbolic quaternions* that are used in Section 6. Then a negative-valued parameter inverts the signs of certain specific products associated with that parameter in the multiplication table of the ungeneralized hyperbolic quaternions $\mathbb{M}$, while a positive-valued parameter causes no sign inversions. With this in mind, we define the *parameterized conjugation operator* $C(\bullet)$ which is simply the imaginary part multiplication operator with its parameter restricted to the two values $+1$ and -1.

Definition 5.6 (Parameterized Conjugation Operator)**.** We define the *parameterized conjugation operator* $C(\bullet)$ by the same symbol and superscript convention as the imaginary part multiplication operator of Definition 5.3. For some parameter, say γ, which must be restricted to *only* the two values $+1$ and -1, it operates on the element w of the next lower (elliptic or hyperbolic) Cayley-Dickson algebra by multiplying the imaginary part of w by γ as in

$$w^{C(\gamma)} := \frac{1}{2}\Big(\big(w + \overline{w}\big) + \gamma\big(w - \overline{w}\big)\Big) = \operatorname{Re} w + \gamma \operatorname{Im} w. \tag{5.9}$$

Note that the parameterized conjugation operator $C(\bullet)$ acts on any element w as in $w^{C(\gamma)}$, to conjugate w, if $\gamma = -1$, or to not conjugate it, if $\gamma = +1$. Of course, for an expression such as $\overline{w}^{C(\gamma)}$ the net effect is reversed.

Starting with the ultrageneralized hyperbolic quaternions, a particular choice of values for the parameters α, β, γ, and δ may be represented by the ordered 4-tuple $(\alpha, \beta, \gamma, \delta)$. For example, the split hyperbolic quaternions $\mathbb{M}\!\!\!\mathbb{M}$ are specified as generated from the double numbers $\mathbb{D}$ by the level 2 parameter 4-tuple $(-1, +1, +1, +1)$.

Let us now restrict our considerations to the standard, nondegenerate, hyperbolic Cayley-Dickson algebras, whose parameters all take values from the set $\{+1, -1\}$. Let us also reverse the order of the parameters in the 4-tuple $(\alpha, \beta, \gamma, \delta)$ and translate the reversed 4-tuple into a sequence of binary digits according to the correspondence $+1 \mapsto 0$ and $-1 \mapsto 1$. When the resulting sequence of binary digits is interpreted as a 4-digit binary number and replaced by its decimal equivalent any parameter 4-tuple $(\alpha, \beta, \gamma, \delta)$ becomes equivalent to a single integer from 0 to 15 in one-to-one correspondence.

Thus, consistent with Definition 1.2 of the split elliptic and hyperbolic Cayley-Dickson algebras given in the Introduction, we may use this one-to-one correspondence between parameter 4-tuples and the integers $0, 1, \ldots, 15$ to establish the following definition.

Definition 5.7 (The Split, Standard, Nondegenerate, Ultrageneralized Hyperbolic n-Level Cayley-Dickson Algebras)**.** A *split, standard, nondegenerate, ultrageneralized hyperbolic n-level Cayley-Dickson algebra* (or, for short, *split ultrageneralized hyperbolic n-level Cayley-Dickson algebra*) is any standard, nondegenerate, ultrageneralized hyperbolic n-level Cayley-Dickson algebra with an odd integer representing its parameter 4-tuple at level n, and even integers representing its parameter 4-tuples at all other levels.

Furthermore, using this convention we may adopt a notation similar to Lounesto's in Reference [24, p. 285] and our own $ECD(\gamma_1, \gamma_2, \dots \gamma_n)$ used in the paragraph following Definition 1.1. Therefore, we may indicate the sequence of parameter 4-tuples defining a standard, nondegenerate, ultrageneralized hyperbolic Cayley-Dickson algebra at any level n by an n-tuple of non-negative integers appended to the symbol $UHCD$. At levels $n \geq 2$ these numbers may take any value in the range from 0 to 15. However, as mentioned earlier, at level 1 only the parameter α remains in Equation (5.7), with the values of β, γ, and δ becoming irrelevant. Thus, the set of numbers representing the 16 allowed parameter choices at higher levels reduces at level 1 to just the two smallest, namely, 0 for the algebra $\mathbb{D}$ ($\cong \mathbb{C}\!\!\!\backslash$), and 1 for the algebra $\mathbb{D}\!\!\!\backslash$ ($\cong \mathbb{C}$). Then, for example, the split hyperbolic quaternions are represented as $\mathbb{M}\!\!\!\backslash \cong UHCD(0,1)$; the hyperbolic octonions, as $\mathbb{G} \cong UHCD(0,0,0)$; and the split hyperbolic octonions, as $\mathbb{G}\!\!\!\backslash \cong UHCD(0,0,1)$.

However, in practice, at level 2 of the standard, nondegenerate, ultrageneralized hyperbolic quaternions, which are fundamental to constructing the orientation congruent algebras in the following section, it is easier to represent the parameter choices by subscripts of strings of letters from the set $\{\, \alpha, \beta, \gamma, \delta \,\}$ appended to the base symbol $\mathbb{M}$ with each letter appearing at most once. Thus, for example, we have $\mathbb{M} \cong UHCD(0,0)$, $\mathbb{M}\!\!\!\backslash \cong \mathbb{M}_\alpha \cong UHCD(0,1)$, and $\mathbb{M}_{\alpha\beta} \cong UHCD(0,3)$. Moreover, due to the fundamental importance of the split ultrageneralized hyperbolic quaternions in representing the orientation congruent algebras, we prefer to keep the backslashed notation rather than employ the letter α as a subscript. Thus, for example, we have the following isomorphisms $\mathbb{M}\!\!\!\backslash_\beta \cong \mathbb{M}_{\alpha\beta} \cong UHCD(0,3)$.

6. The Duality Classification of Algebras

NOTE: Most of this section is wrong and needs much rewriting. Nevertheless, there is a structure to the tensor products of algebras that represent and classify the orientation congruent algebras. I am currently investigating it and have some preliminary notes I am unable to incorporate into this draft now. Hints of it are given at the end of the last section.

Table 6.3 presents a shortened version of the standard Clifford algebra classification in terms of the matrix algebras over the real and complex numbers, and the quaternions. The abbreviated notation used here is found in Lounesto's [24, Ch. 16] and Porteous' [33], [34], [35, Ch. 15], [36] works. Explaining it by example, the direct sum of the quaternion algebra with itself, $\mathbb{H} \oplus \mathbb{H}$, becomes ${}^2\mathbb{H}$ in Porteous' notation, while the full 2×2 matrix algebra over the quaternions, sometimes written as $\mathrm{Mat}(\mathbb{H}, 2)$, becomes $\mathbb{H}(2)$.

Table 6.4 presents a shortened version of Keller's Clifford algebra classification in terms of the matrix algebras over the real, complex, and double numbers, and the quaternions. The Zentralblatt reviewer [19] dismisses Keller's use of the double numbers (or *duplex algebra* in Keller's words) to classify the Clifford algebras) by saying, "Using the elementary fact that $\mathbb{D} \cong \mathbb{R} \oplus \mathbb{R}$ as a ring, the author makes a trivial notational amendment to the table of Clifford algebras." However, it is in the dual relationship of the Clifford and orientation congruent algebras that the seed of Keller's notation becomes alive and grows to maturity.

Tables 6.5 and 6.6 present shortened versions of the new duality classification of the Clifford and orientation congruent algebras. Tables 6.8 and 6.9 present full versions (up to mod 8 periodicity) of the duality classification of the Clifford and orientation congruent algebras. The notation in the full tables is abbreviated by dropping the tensor product symbol $\otimes$, so that, for example, $\mathbb{D} \otimes \not\mathbb{H}^{\otimes 2}$ becomes $\mathbb{D}\,\not\mathbb{H}^2$.

The publications of Mosna et al. [30, p. 4401] and Porteous [36, p. 35] give the following useful isomorphisms (although we have added the last tensor products in the first two chains)

$$\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \cong \mathbb{C} \otimes \mathbb{D}, \quad \mathbb{C} \otimes \mathbb{H} \cong \mathbb{C}(2) \cong \mathbb{C} \otimes \mathbb{R}(2), \quad \text{and } \mathbb{H} \otimes \mathbb{H} \cong \mathbb{R}(4).$$

Also note that $\mathbb{R}(2)$ is isomorphic to $\not\mathbb{H}$. This fact is used extensively in constructing the duality classifications.

The duality between the Clifford and orientation congruent algebras is expressed in Tables 6.5, 6.6, 6.8, and 6.9 in the following way. Except for cases of $n = p + q = 0, 1$ comprising the self-dual algebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{D}$, the orientation congruent algebra $\mathcal{OC}_{q,p}$ is the algebra dual of the Clifford algebra $\mathcal{C}\ell_{p,q}$.

The tensor products of algebras representing these algebras are also in duality. One such representation is transformed into its dual not by altering any real, complex, or double algebra symbols appearing in its tensor product, but by swapping any hyperbolic quaternion algebra, ungeneralized or split, for its corresponding elliptic quaternion algebra and vice versa. In symbols this is $\mathbb{R} \mapsto \mathbb{R}$, $\mathbb{C} \mapsto \mathbb{C}$, $\mathbb{D} \mapsto \mathbb{D}$, $\mathbb{H} \mapsto \mathbb{M}$, $\mathbb{M} \mapsto \mathbb{H}$, $\not\mathbb{H} \mapsto \not\mathbb{M}$, and $\not\mathbb{M} \mapsto \not\mathbb{H}$. Thus, except for the $n = 0, 1$ cases, the sign of the quantity $p - q$ must be inverted mod 8 to get a hyperbolic interpretation of Okubo's classification in Table 6.1, as well as the duality classification in Table 6.2, that can be applied to the orientation congruent algebra.

Even though in the usual elliptic vs. hyperbolic sense the algebras $\mathbb{C}$ and $\mathbb{D}$ are duals, in the $\mathcal{C}\ell$ vs. $\mathcal{OC}$ algebra duality sense they, along with $\mathbb{R}$, are self-duals. That is because of the duality-invariant structure-determining role of the algebras $\mathbb{C}$ and $\mathbb{D}$. In this way duality relationships are determined solely by the ungeneralized or split elliptic (resp., hyperbolic) quaternion algebras of the tensor product representing a Clifford (resp., orientation congruent) algebra.

Let $n = p + q$ be called the *base dimension* of the $\mathcal{C}\ell_{p,q}$ and $\mathcal{OC}_{p,q}$ algebras. Then the split elliptic (resp., hyperbolic) quaternionic elements of the tensor product representation also contribute to the *total dimension* 2^n of the algebra $\mathcal{C}\ell_{p,q}$ (resp., $\mathcal{OC}_{p,q}$) by varying with the base dimension through the factors $4^{n/2} = 2^n$ and $4^{(n-2)/2} = 2^{n-2}$ derived, respectively, from the tensor products $\mathbb{H}^{\otimes n/2}$ and $\mathbb{H}^{\otimes(n-2)/2}$ (resp., $\mathbb{M}^{\otimes n/2}$ and $\mathbb{M}^{\otimes(n-2)/2}$) for even n, and the factors $4^{(n-1)/2} = 2^{n-1}$ and $4^{(n-3)/2} = 2^{n-3}$ derived, respectively, from the tensor products $\mathbb{H}^{\otimes(n-1)/2}$ and $\mathbb{H}^{\otimes(n-3)/2}$ (resp., $\mathbb{M}^{\otimes(n-1)/2}$ and $\mathbb{M}^{\otimes(n-3)/2}$) for odd n.

The duality classification of the Clifford and orientation congruent algebras adds two more fundamental classifications to the three of Okubo's classification scheme in Table 6.1. This is because his *normal* class splits into the new *real* and *double* classes, while Okubo's *quaternionic* class splits into the new *singly quaternionic* and *doubly quaternionic* classes. For simplicity and consistency with the designations just given, in the duality classification we rename his *almost complex* case to *complex*. The duality classification is presented in Table 6.2.

Table 6.1. Susumu Okubo's classification of the real Clifford algebras [31, 32]. This classification is determined by the column index $p - q$ of the following tables taken mod 8. The cases marked with an asterisk admit two inequivalent irreducible real matrix realizations; the other cases have unique irreducible realizations.

Classification	$p-q$ mod 8	
	odd $n = p+q$	even $n = p+q$
normal	1*	0 or 2
almost complex (a.c.)	3 or 7	
quaternionic (quat.)	5*	4 or 6

Table 6.2. The duality classification of the real Clifford algebras. This classification is determined by the column index $p - q$ of the following tables taken mod 8. The cases marked with an asterisk admit two inequivalent irreducible real matrix realizations; the other cases have unique irreducible realizations.

Classification	$p-q$ mod 8	
	odd $n = p+q$	even $n = p+q$
real		0 or 2
double	1*	
complex	3 or 7	
singly quaternionic		4 or 6
doubly quaternionic	5*	

TABLE 6.3. Shortened table of the standard classification of the Clifford algebras $\mathcal{C}\ell_{p,q}$ in Porteous' notation. See, for example, [24, Ch. 16], [35, Ch. 15] or [36].

$p+q$ \ $p-q$	-3	-2	-1	0	1	2	3
0				$\mathbb{R}$			
1			$\mathbb{C}$		${}^2\mathbb{R}$		
2		$\mathbb{H}$		$\mathbb{R}(2)$		$\mathbb{R}(2)$	
3	${}^2\mathbb{H}$		$\mathbb{C}(2)$		${}^2\mathbb{R}(2)$		$\mathbb{C}(2)$

TABLE 6.4. Keller's classification of the Clifford algebras $\mathcal{C}\ell_{p,q}$, a shortened version of his Table III in Reference [19].

$p+q$ \ $p-q$	-3	-2	-1	0	1	2	3
0				$\mathbb{R}(1)$			
1			$\mathbb{C}(1)$		$\mathbb{D}(1)$		
2		$\mathbb{H}(1)$		$\mathbb{R}(2)$		$\mathbb{R}(2)$	
3	${}^2\mathbb{H}(1)$		$\mathbb{C}(2)$		$\mathbb{D}(2)$		$\mathbb{C}(2)$

TABLE 6.5. Shortened table of the duality classification of the Clifford algebras $\mathcal{C}\ell_{p,q}$.

$p+q$ \ $p-q$	-3	-2	-1	0	1	2	3
0				$\mathbb{R}$			
1			$\mathbb{C}$		$\mathbb{D}$		
2		$\mathbb{H}$		$\not\mathbb{H}$		$\not\mathbb{H}$	
3	$\mathbb{D}\otimes\mathbb{H}$		$\mathbb{C}\otimes\not\mathbb{H}$		$\mathbb{D}\otimes\not\mathbb{H}$		$\mathbb{C}\otimes\not\mathbb{H}$

TABLE 6.6. Shortened table of the duality classification of the orientation congruent algebras $\mathcal{OC}_{p,q}$.

$p+q$ \ $p-q$	-3	-2	-1	0	1	2	3
0				$\mathbb{R}$			
1			$\mathbb{C}$		$\mathbb{D}$		
2		$\not\mathbb{M}$		$\not\mathbb{M}$		$\mathbb{M}$	
3	$\mathbb{C}\otimes\not\mathbb{M}$		$\mathbb{D}\otimes\not\mathbb{M}$		$\mathbb{C}\otimes\not\mathbb{M}$		$\mathbb{D}\otimes\mathbb{M}$

TABLE 6.7. The standard classification of the Clifford algebras $\mathcal{C}\ell_{p,q}$ in Porteous' notation. See, for example, [24, Ch. 16], [35, Ch. 15] or [36].

$p-q$ mod 8	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
$p+q$ \ $p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$\mathbb{R}$							
1							$\mathbb{C}$		${}^2\mathbb{R}$						
2						$\mathbb{H}$		$\mathbb{R}(2)$		$\mathbb{R}(2)$					
3					${}^2\mathbb{H}$		$\mathbb{C}(2)$		${}^2\mathbb{R}(2)$		$\mathbb{C}(2)$				
4				$\mathbb{H}(2)$		$\mathbb{H}(2)$		$\mathbb{R}(4)$		$\mathbb{R}(4)$		$\mathbb{H}(2)$			
5			$\mathbb{C}(4)$		${}^2\mathbb{H}(2)$		$\mathbb{C}(4)$		${}^2\mathbb{R}(4)$		$\mathbb{C}(4)$		${}^2\mathbb{H}(2)$		
6		$\mathbb{R}(8)$		$\mathbb{H}(4)$		$\mathbb{H}(4)$		$\mathbb{R}(8)$		$\mathbb{R}(8)$		$\mathbb{H}(4)$		$\mathbb{H}(4)$	
7	${}^2\mathbb{R}(8)$		$\mathbb{C}(8)$		${}^2\mathbb{H}(4)$		$\mathbb{C}(8)$		${}^2\mathbb{R}(8)$		$\mathbb{C}(8)$		${}^2\mathbb{H}(4)$		$\mathbb{C}(8)$
even n		$\mathbb{R}\left(2^{\frac{n}{2}}\right)$		$\mathbb{H}\left(2^{\frac{n-2}{2}}\right)$		$\mathbb{H}\left(2^{\frac{n-2}{2}}\right)$		$\mathbb{R}\left(2^{\frac{n}{2}}\right)$		$\mathbb{R}\left(2^{\frac{n}{2}}\right)$		$\mathbb{H}\left(2^{\frac{n-2}{2}}\right)$		$\mathbb{H}\left(2^{\frac{n-2}{2}}\right)$	
odd n	${}^2\mathbb{R}\left(2^{\frac{n-1}{2}}\right)$		$\mathbb{C}\left(2^{\frac{n-1}{2}}\right)$		${}^2\mathbb{H}\left(2^{\frac{n-3}{2}}\right)$		$\mathbb{C}\left(2^{\frac{n-1}{2}}\right)$		${}^2\mathbb{R}\left(2^{\frac{n-1}{2}}\right)$		$\mathbb{C}\left(2^{\frac{n-1}{2}}\right)$		${}^2\mathbb{H}\left(2^{\frac{n-3}{2}}\right)$		$\mathbb{C}\left(2^{\frac{n-1}{2}}\right)$

TABLE 6.8. The duality classification of the Clifford algebras $\mathcal{C}\ell_{p,q}$.

$p-q$ mod 8	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
$p+q$ \ $p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$\mathbb{R}$							
1							$\mathbb{C}$		$\mathbb{D}$						
2						$\mathbb{H}$		$\mathbb{N}$		$\mathbb{N}$					
3					$\mathbb{D}\,\mathbb{H}$		$\mathbb{C}\,\mathbb{N}$		$\mathbb{D}\,\mathbb{N}$		$\mathbb{C}\,\mathbb{N}$				
4				$\mathbb{H}\,\mathbb{N}$		$\mathbb{H}\,\mathbb{N}$		$\mathbb{N}^2$		$\mathbb{N}^2$		$\mathbb{H}\,\mathbb{N}$			
5			$\mathbb{C}\,\mathbb{N}^2$		$\mathbb{D}\,\mathbb{H}\,\mathbb{N}$		$\mathbb{C}\,\mathbb{N}^2$		$\mathbb{D}\,\mathbb{N}^2$		$\mathbb{C}\,\mathbb{N}^2$		$\mathbb{D}\,\mathbb{H}\,\mathbb{N}$		
6		$\mathbb{N}^3$		$\mathbb{H}\,\mathbb{N}^2$		$\mathbb{H}\,\mathbb{N}^2$		$\mathbb{N}^3$		$\mathbb{N}^3$		$\mathbb{H}\,\mathbb{N}^2$		$\mathbb{H}\,\mathbb{N}^2$	
7	$\mathbb{D}\,\mathbb{N}^3$		$\mathbb{C}\,\mathbb{N}^3$		$\mathbb{D}\,\mathbb{H}\,\mathbb{N}^2$		$\mathbb{C}\,\mathbb{N}^3$		$\mathbb{D}\,\mathbb{N}^3$		$\mathbb{C}\,\mathbb{N}^3$		$\mathbb{D}\,\mathbb{H}\,\mathbb{N}^2$		$\mathbb{C}\,\mathbb{N}^3$
even n		$\mathbb{N}^{\frac{n}{2}}$		$\mathbb{H}\,\mathbb{N}^{\frac{n-2}{2}}$		$\mathbb{H}\,\mathbb{N}^{\frac{n-2}{2}}$		$\mathbb{N}^{\frac{n}{2}}$		$\mathbb{N}^{\frac{n}{2}}$		$\mathbb{H}\,\mathbb{N}^{\frac{n-2}{2}}$		$\mathbb{H}\,\mathbb{N}^{\frac{n-2}{2}}$	
odd n	$\mathbb{D}\,\mathbb{N}^{\frac{n-1}{2}}$		$\mathbb{C}\,\mathbb{N}^{\frac{n-1}{2}}$		$\mathbb{D}\,\mathbb{H}\,\mathbb{N}^{\frac{n-3}{2}}$		$\mathbb{C}\,\mathbb{N}^{\frac{n-1}{2}}$		$\mathbb{D}\,\mathbb{N}^{\frac{n-1}{2}}$		$\mathbb{C}\,\mathbb{N}^{\frac{n-1}{2}}$		$\mathbb{D}\,\mathbb{H}\,\mathbb{N}^{\frac{n-3}{2}}$		$\mathbb{C}\,\mathbb{N}^{\frac{n-1}{2}}$

TABLE 6.9. The duality classification of the orientation congruent algebras $\mathcal{OC}_{p,q}$.

$p-q$ mod 8	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
$p+q$ \ $p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$\mathbb{R}$							
1							$\mathbb{C}$		$\mathbb{D}$						
2						$\mathbb{M}$		$\mathbb{M}$		$\mathbb{M}$					
3					$\mathbb{C}\mathbb{M}$		$\mathbb{D}\mathbb{M}$		$\mathbb{C}\mathbb{M}$		$\mathbb{D}\mathbb{M}$				
4				$\mathbb{M}\mathbb{M}$		$\mathbb{M}^2$		$\mathbb{M}^2$		$\mathbb{M}\mathbb{M}$		$\mathbb{M}\mathbb{M}$			
5			$\mathbb{D}\mathbb{M}\mathbb{M}$		$\mathbb{C}\mathbb{M}^2$		$\mathbb{D}\mathbb{M}^2$		$\mathbb{C}\mathbb{M}^2$		$\mathbb{D}\mathbb{M}\mathbb{M}$		$\mathbb{C}\mathbb{M}^2$		
6		$\mathbb{M}\mathbb{M}^2$		$\mathbb{M}\mathbb{M}^2$		$\mathbb{M}^3$		$\mathbb{M}^3$		$\mathbb{M}\mathbb{M}^2$		$\mathbb{M}\mathbb{M}^2$		$\mathbb{M}^3$	
7	$\mathbb{C}\mathbb{M}^3$		$\mathbb{D}\mathbb{M}\mathbb{M}^2$		$\mathbb{C}\mathbb{M}^3$		$\mathbb{D}\mathbb{M}^3$		$\mathbb{C}\mathbb{M}^3$		$\mathbb{D}\mathbb{M}\mathbb{M}^2$		$\mathbb{C}\mathbb{M}^3$		$\mathbb{D}\mathbb{M}^3$
even n		$\mathbb{M}\mathbb{M}^{\frac{n-2}{2}}$		$\mathbb{M}\mathbb{M}^{\frac{n-2}{2}}$		$\mathbb{M}^{\frac{n}{2}}$		$\mathbb{M}^{\frac{n}{2}}$		$\mathbb{M}\mathbb{M}^{\frac{n-2}{2}}$		$\mathbb{M}\mathbb{M}^{\frac{n-2}{2}}$		$\mathbb{M}^{\frac{n}{2}}$	
odd n	$\mathbb{C}\mathbb{M}^{\frac{n-1}{2}}$		$\mathbb{D}\mathbb{M}\mathbb{M}^{\frac{n-3}{2}}$		$\mathbb{C}\mathbb{M}^{\frac{n-1}{2}}$		$\mathbb{D}\mathbb{M}^{\frac{n-1}{2}}$		$\mathbb{C}\mathbb{M}^{\frac{n-1}{2}}$		$\mathbb{D}\mathbb{M}\mathbb{M}^{\frac{n-3}{2}}$		$\mathbb{C}\mathbb{M}^{\frac{n-1}{2}}$		$\mathbb{D}\mathbb{M}^{\frac{n-1}{2}}$

Table 6.10. The tilt (signature inversion) symmetry of the Clifford algebras $\mathcal{C}\ell_{p,q}$.

$p-q$ mod 8	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
$p+q$ \ $p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$\mathbb{R}$							
1							$(\mathbb{C})$		$(\bcancel{\mathbb{C}})$						
2						$(\mathbb{H})$		$\bcancel{\mathbb{H}}$		$(\bcancel{\mathbb{H}})$					
3					$(\bcancel{\mathbb{C}}\mathbb{H})$		$(\mathbb{C})\bcancel{\mathbb{H}}$		$(\bcancel{\mathbb{C}})\bcancel{\mathbb{H}}$		$(\mathbb{C}\bcancel{\mathbb{H}})$				
4				$(\mathbb{H})\bcancel{\mathbb{H}}$		$(\mathbb{H})\bcancel{\mathbb{H}}$		$\bcancel{\mathbb{H}}^2$		$(\bcancel{\mathbb{H}})\bcancel{\mathbb{H}}$		$(\mathbb{H})\bcancel{\mathbb{H}}$			
5			$(\mathbb{C}\bcancel{\mathbb{H}})\bcancel{\mathbb{H}}$		$(\bcancel{\mathbb{C}}\mathbb{H})\bcancel{\mathbb{H}}$		$(\mathbb{C})\bcancel{\mathbb{H}}^2$		$(\bcancel{\mathbb{C}})\bcancel{\mathbb{H}}^2$		$(\mathbb{C}\bcancel{\mathbb{H}})\bcancel{\mathbb{H}}$		$(\bcancel{\mathbb{C}}\mathbb{H})\bcancel{\mathbb{H}}$		
6		$(\bcancel{\mathbb{H}})\bcancel{\mathbb{H}}^2$		$(\mathbb{H})\bcancel{\mathbb{H}}^2$		$(\mathbb{H})\bcancel{\mathbb{H}}^2$		$\bcancel{\mathbb{H}}^3$		$(\bcancel{\mathbb{H}})\bcancel{\mathbb{H}}^2$		$(\mathbb{H})\bcancel{\mathbb{H}}^2$		$(\mathbb{H})\bcancel{\mathbb{H}}^2$	
7	$(\bcancel{\mathbb{C}})\bcancel{\mathbb{H}}^3$		$(\mathbb{C}\bcancel{\mathbb{H}})\bcancel{\mathbb{H}}^2$		$(\bcancel{\mathbb{C}}\mathbb{H})\bcancel{\mathbb{H}}^2$		$(\mathbb{C})\bcancel{\mathbb{H}}^3$		$(\bcancel{\mathbb{C}})\bcancel{\mathbb{H}}^3$		$(\mathbb{C}\bcancel{\mathbb{H}})\bcancel{\mathbb{H}}^2$		$(\bcancel{\mathbb{C}}\mathbb{H})\bcancel{\mathbb{H}}^2$		$(\mathbb{C})\bcancel{\mathbb{H}}^3$
even n		$(\bcancel{\mathbb{H}})\bcancel{\mathbb{H}}^{\frac{n-2}{2}}$		$(\mathbb{H})\bcancel{\mathbb{H}}^{\frac{n-2}{2}}$		$(\mathbb{H})\bcancel{\mathbb{H}}^{\frac{n-2}{2}}$		$\bcancel{\mathbb{H}}^{\frac{n}{2}}$		$(\bcancel{\mathbb{H}})\bcancel{\mathbb{H}}^{\frac{n-2}{2}}$		$(\mathbb{H})\bcancel{\mathbb{H}}^{\frac{n-2}{2}}$		$(\mathbb{H})\bcancel{\mathbb{H}}^{\frac{n-2}{2}}$	
odd n	$(\bcancel{\mathbb{C}})\bcancel{\mathbb{H}}^{\frac{n-1}{2}}$		$(\mathbb{C}\bcancel{\mathbb{H}})\bcancel{\mathbb{H}}^{\frac{n-3}{2}}$		$(\bcancel{\mathbb{C}}\mathbb{H})\bcancel{\mathbb{H}}^{\frac{n-3}{2}}$		$(\mathbb{C})\bcancel{\mathbb{H}}^{\frac{n-1}{2}}$		$(\bcancel{\mathbb{C}})\bcancel{\mathbb{H}}^{\frac{n-1}{2}}$		$(\mathbb{C}\bcancel{\mathbb{H}})\bcancel{\mathbb{H}}^{\frac{n-3}{2}}$		$(\bcancel{\mathbb{C}}\mathbb{H})\bcancel{\mathbb{H}}^{\frac{n-3}{2}}$		$(\mathbb{C})\bcancel{\mathbb{H}}^{\frac{n-1}{2}}$

Table 6.11. The tilt (signature inversion) symmetry of the orientation congruent algebras $\mathcal{OC}_{p,q}$.

$p-q \mod 8$	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
$p+q$ \ $p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$\mathbb{R}$							
1							$(\mathbb{D})$		$(\mathbb{D})$						
2						$(\mathbb{M})$		$\mathbb{M}$		$(\mathbb{M})$					
3					$(\mathbb{D}\mathbb{M})$		$(\mathbb{D})\mathbb{M}$		$(\mathbb{D})\mathbb{M}$		$(\mathbb{D}\mathbb{M})$				
4				$(\mathbb{M})\mathbb{M}$		$(\mathbb{M})\mathbb{M}$		$\mathbb{M}^2$		$(\mathbb{M})\mathbb{M}$		$(\mathbb{M})\mathbb{M}$			
5			$(\mathbb{D}\mathbb{M})\mathbb{M}$		$(\mathbb{D}\mathbb{M})\mathbb{M}$		$(\mathbb{D})\mathbb{M}^2$		$(\mathbb{D})\mathbb{M}^2$		$(\mathbb{D}\mathbb{M})\mathbb{M}$		$(\mathbb{D}\mathbb{M})\mathbb{M}$		
6		$(\mathbb{M})\mathbb{M}^2$		$(\mathbb{M})\mathbb{M}^2$		$(\mathbb{M})\mathbb{M}^2$		$\mathbb{M}^3$		$(\mathbb{M})\mathbb{M}^2$		$(\mathbb{M})\mathbb{M}^2$		$(\mathbb{M})\mathbb{M}^2$	
7	$(\mathbb{D})\mathbb{M}^3$		$(\mathbb{D}\mathbb{M})\mathbb{M}^2$		$(\mathbb{D}\mathbb{M})\mathbb{M}^2$		$(\mathbb{D})\mathbb{M}^3$		$(\mathbb{D})\mathbb{M}^3$		$(\mathbb{D}\mathbb{M})\mathbb{M}^2$		$(\mathbb{D}\mathbb{M})\mathbb{M}^2$		$(\mathbb{D})\mathbb{M}^3$
even n		$(\mathbb{M})\mathbb{M}^{\frac{n-2}{2}}$		$(\mathbb{M})\mathbb{M}^{\frac{n-2}{2}}$		$(\mathbb{M})\mathbb{M}^{\frac{n-2}{2}}$		$\mathbb{M}^{\frac{n}{2}}$		$(\mathbb{M})\mathbb{M}^{\frac{n-2}{2}}$		$(\mathbb{M})\mathbb{M}^{\frac{n-2}{2}}$		$(\mathbb{M})\mathbb{M}^{\frac{n-2}{2}}$	
odd n	$(\mathbb{D})\mathbb{M}^{\frac{n-1}{2}}$		$(\mathbb{D}\mathbb{M})\mathbb{M}^{\frac{n-3}{2}}$		$(\mathbb{D}\mathbb{M})\mathbb{M}^{\frac{n-3}{2}}$		$(\mathbb{D})\mathbb{M}^{\frac{n-1}{2}}$		$(\mathbb{D})\mathbb{M}^{\frac{n-1}{2}}$		$(\mathbb{D}\mathbb{M})\mathbb{M}^{\frac{n-3}{2}}$		$(\mathbb{D}\mathbb{M})\mathbb{M}^{\frac{n-3}{2}}$		$(\mathbb{D})\mathbb{M}^{\frac{n-1}{2}}$

Table 6.12. The totally dual classification of the Clifford algebras $\mathcal{C}\ell_{p,q}$.

$p-q \mod 8$	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
$p+q$ \ $p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$\mathbb{R}$							
1							$(\mathbb{C})$		$(\not\mathbb{C})$						
2						$(\mathbb{H})$		$(\not\mathbb{H})$		$(\not\mathbb{H})$					
3					$(\not\mathbb{C}\mathbb{H})$		$(\mathbb{C}\mathbb{H})$		$(\not\mathbb{C}\not\mathbb{H})$		$(\mathbb{C}\not\mathbb{H})$				
4				$(\mathbb{H})\not\mathbb{H}$		$(\mathbb{H})\not\mathbb{H}$		$(\not\mathbb{H})\not\mathbb{H}$		$(\not\mathbb{H})\not\mathbb{H}$		$(\mathbb{H})\not\mathbb{H}$			
5			$(\mathbb{C}\not\mathbb{H})\not\mathbb{H}$		$(\not\mathbb{C}\mathbb{H})\not\mathbb{H}$		$(\mathbb{C}\mathbb{H})\not\mathbb{H}$		$(\not\mathbb{C}\not\mathbb{H})\not\mathbb{H}$		$(\mathbb{C}\not\mathbb{H})\not\mathbb{H}$		$(\not\mathbb{C}\mathbb{H})\not\mathbb{H}$		
6		$(\not\mathbb{H})\not\mathbb{H}^2$		$(\mathbb{H})\not\mathbb{H}^2$		$(\mathbb{H})\not\mathbb{H}^2$		$(\not\mathbb{H})\not\mathbb{H}^2$		$(\not\mathbb{H})\not\mathbb{H}^2$		$(\mathbb{H})\not\mathbb{H}^2$		$(\mathbb{H})\not\mathbb{H}^2$	
7	$(\not\mathbb{C}\not\mathbb{H})\not\mathbb{H}^2$		$(\mathbb{C}\not\mathbb{H})\not\mathbb{H}^2$		$(\not\mathbb{C}\mathbb{H})\not\mathbb{H}^2$		$(\mathbb{C}\mathbb{H})\not\mathbb{H}^2$		$(\not\mathbb{C}\not\mathbb{H})\not\mathbb{H}^2$		$(\mathbb{C}\not\mathbb{H})\not\mathbb{H}^2$		$(\not\mathbb{C}\mathbb{H})\not\mathbb{H}^2$		$(\mathbb{C}\mathbb{H})\not\mathbb{H}^2$
even n		$(\not\mathbb{H})\not\mathbb{H}^{\frac{n-2}{2}}$		$(\mathbb{H})\not\mathbb{H}^{\frac{n-2}{2}}$		$(\mathbb{H})\not\mathbb{H}^{\frac{n-2}{2}}$		$(\not\mathbb{H})\not\mathbb{H}^{\frac{n-2}{2}}$		$(\not\mathbb{H})\not\mathbb{H}^{\frac{n-2}{2}}$		$(\mathbb{H})\not\mathbb{H}^{\frac{n-2}{2}}$		$(\mathbb{H})\not\mathbb{H}^{\frac{n-2}{2}}$	
odd n	$(\not\mathbb{C}\not\mathbb{H})\not\mathbb{H}^{\frac{n-3}{2}}$		$(\mathbb{C}\not\mathbb{H})\not\mathbb{H}^{\frac{n-3}{2}}$		$(\not\mathbb{C}\mathbb{H})\not\mathbb{H}^{\frac{n-3}{2}}$		$(\mathbb{C}\mathbb{H})\not\mathbb{H}^{\frac{n-3}{2}}$		$(\not\mathbb{C}\not\mathbb{H})\not\mathbb{H}^{\frac{n-3}{2}}$		$(\mathbb{C}\not\mathbb{H})\not\mathbb{H}^{\frac{n-3}{2}}$		$(\not\mathbb{C}\mathbb{H})\not\mathbb{H}^{\frac{n-3}{2}}$		$(\mathbb{C}\mathbb{H})\not\mathbb{H}^{\frac{n-3}{2}}$

TABLE 6.13. The totally dual classification of the orientation congruent algebras $\mathcal{OC}_{p,q}$.

$p-q \mod 8$	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
$p+q$ \ $p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$\mathbb{R}$							
1							$(\mathbb{D})$		$(\mathbb{D})$						
2						$(\mathbb{M})$		$(\mathbb{M})$		$(\mathbb{M})$					
3					$(\mathbb{DM})$		$(\mathbb{DM})$		$(\mathbb{DM})$		$(\mathbb{DM})$				
4				$(\mathbb{M})\mathbb{M}$		$(\mathbb{M})\mathbb{M}$		$(\mathbb{M})\mathbb{M}$		$(\mathbb{M})\mathbb{M}$		$(\mathbb{M})\mathbb{M}$			
5			$(\mathbb{DM})\mathbb{M}$		$(\mathbb{DM})\mathbb{M}$		$(\mathbb{DM})\mathbb{M}$		$(\mathbb{DM})\mathbb{M}$		$(\mathbb{DM})\mathbb{M}$		$(\mathbb{DM})\mathbb{M}$		
6		$(\mathbb{M})\mathbb{M}^2$		$(\mathbb{M})\mathbb{M}^2$		$(\mathbb{M})\mathbb{M}^2$		$(\mathbb{M})\mathbb{M}^2$		$(\mathbb{M})\mathbb{M}^2$		$(\mathbb{M})\mathbb{M}^2$		$(\mathbb{M})\mathbb{M}^2$	
7	$(\mathbb{DM})\mathbb{M}^2$		$(\mathbb{DM})\mathbb{M}^2$		$(\mathbb{DM})\mathbb{M}^2$		$(\mathbb{DM})\mathbb{M}^2$		$(\mathbb{DM})\mathbb{M}^2$		$(\mathbb{DM})\mathbb{M}^2$		$(\mathbb{DM})\mathbb{M}^2$		$(\mathbb{DM})\mathbb{M}^2$
even n		$(\mathbb{M})\mathbb{M}^{\frac{n-2}{2}}$		$(\mathbb{M})\mathbb{M}^{\frac{n-2}{2}}$		$(\mathbb{M})\mathbb{M}^{\frac{n-2}{2}}$		$(\mathbb{M})\mathbb{M}^{\frac{n-2}{2}}$		$(\mathbb{M})\mathbb{M}^{\frac{n-2}{2}}$		$(\mathbb{M})\mathbb{M}^{\frac{n-2}{2}}$		$(\mathbb{M})\mathbb{M}^{\frac{n-2}{2}}$	
odd n	$(\mathbb{DM})\mathbb{M}^{\frac{n-3}{2}}$		$(\mathbb{DM})\mathbb{M}^{\frac{n-3}{2}}$		$(\mathbb{DM})\mathbb{M}^{\frac{n-3}{2}}$		$(\mathbb{DM})\mathbb{M}^{\frac{n-3}{2}}$		$(\mathbb{DM})\mathbb{M}^{\frac{n-3}{2}}$		$(\mathbb{DM})\mathbb{M}^{\frac{n-3}{2}}$		$(\mathbb{DM})\mathbb{M}^{\frac{n-3}{2}}$		$(\mathbb{DM})\mathbb{M}^{\frac{n-3}{2}}$

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