The Orientation Congruent Algebra: A Nonassociative Clifford-Like Algebra [1]

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Abstract. The *correlated grade form* of twisted blades (twisted simple multivectors) faithfully renders in symbols their native geometric structure. The discovery of this paper's nonassociative Clifford-like algebra was driven by trying to calculate exterior products of straight and twisted multivectors directly in a basis of this form. The key was found to be the *orientation congruent* ($\mathcal{OC}$) algebra. This paper is being published electronically in about ten sections, each offered as soon as written. In this first section we axiomatize the orientation congruent algebra by generators and relations. The next section derives the *sign factor function* σ and proves that the Clifford product times it is the multiplication of an explicitly Clifford-like algebra isomorphic to the orientation congruent algebra. Later sections are planned to show how to calculate the $\mathcal{OC}$ product in Mathematica and Clical; to define the orientation congruent *contraction operators,* deduce their properties, derive other expressions for them, and use them to compute the $\mathcal{OC}$ product within the exterior algebra using a modified Cartan decomposition formula; to develop the algebra's *product sequence graph* with labeled edges; to derive a *predictor of a null associator* as a function of the grades of the three elements in it; to prove the *associomediative property* of the algebra's *counit;* to develop matrix representations (under a *nonassociative* matrix product) of the orientation congruent product; and to discuss the motivating application per se and as inspiration for the first set of axioms.

Dedication

to Elaine Yaw in honor of friendship

Contents

Preface

In this paper, as is sufficient for an initial development, we limit the base vector space (upon which the larger vector spaces of the exterior, Clifford and orientation congruent algebras may be constructed) to a finite dimension over the reals $\mathbb{R}$. And, although we might easily adopt $\mathbb{R}^n$, the standard notation for a finite-dimensional vector space over $\mathbb{R}$, to symbolize the base vector space we prefer instead to use V^n. The symbol V^n has the disadvantage of requiring the awkward variation $V^n(\mathbb{R})$ to explicitly specify the scalar field. However, we make this choice because of Bossavit's warning[1] that the usual notation confuses a vector space V^n derived from physical modeling with a vector space $\mathbb{R}^n$ that is not only unnecessarily basis-dependent,[2] but also carries topological, metric, or other properties that may or may not apply to V^n.

The superscript in the symbol V^n indicates, of course, the dimension of the base space. Unless otherwise stated, we assume the dimension of V appearing with no superscript to also be n.

Two more common Clifford algebra notations are reserved for other purposes in this paper. Therefore, the *reversion* of a multivector A, sometimes denoted with a tilde as $\widetilde{A}$, is instead symbolized by a superscript dagger as $A^\dagger$. And the *grade involution* of A, often denoted with a hat as $\widehat{A}$, is instead symbolized by a right hooked overline as $\overline{A}\!\rceil$, or by a superscript symbol derived from it, the upper right "corner," as $A^{\urcorner}$.

Also, we use the clear and simple notation $\mathbb{Z}[a, b]$ defined so that for all $a, b \in \mathbb{R}$ it is the inclusive interval of integers $\{\, i \mid i \in \mathbb{Z} \text{ and } a \leq i \leq b \,\}$. Other notations used in this paper that are standard or slightly modified are sometimes explained when introduced; those that are invented for new concepts are, of course, always explained.

To publish this research as quickly as possible this paper will be divided by section into about ten PDF files. As soon as a section is written its file will be available for download from my website `http://felicity.freeshell.org`.

We begin at the heart of this paper, an axiomatic formulation of the orientation congruent algebra, and journey outward. Unfortunately, this order reverses the natural course of development from motivating problem to general principles. But it is the fastest way to make the vital core of these ideas available.

Nevertheless, we briefly mention that the notation used in the application that drove the discovery of the orientation congruent algebra, the *correlated grade form,* is an almost exact symbolic analog of twisted multivectors (as well as twisted multicovectors or multiforms) in their native geometric representation as determined by measurement procedures for the quantities of physical theories. The orientation congruent algebra was specifically developed to calculate with twisted objects on their own terms, so to speak.

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[1] See Ref. [3], p. 2, fn. 3, and some of his other works.

[2] The notation $\mathbb{R}^n$ is also commonly used for the vector space of all n-tuples which are the components of any vector in V^n with respect to some basis.

1 An Axiom System for the Orientation Congruent Algebra

In this section we first discuss the nondegenerate quadratic form $Q_{p,q}$, defining terms and notations for it and the two algebras of our interest associated with it. We then present a deductive foundation for the Clifford algebra $\mathcal{C}\ell_{p,q}$ of a nondegenerate quadratic form $Q_{p,q}$ in terms of *generators of and relations on* its elements(a *GR* axiom system for short). Next we give a similar axiomatic formulation for the orientation congruent algebra $\mathcal{OC}_{p,q}$ that is derived from the one for the corresponding Clifford algebra $\mathcal{C}\ell_{p,q}$ by modifying two of its axioms and adding two new ones. Although we give only GR axiom sets in this first section of the paper, in the penultimate subsection we discuss some alternative axiomatic approaches. Finally, the last subsection presents the multiplication tables for some low order Clifford and orientation congruent algebras.

1.1 The Nondegenerate Quadratic Form $Q_{p,q}$ and Associated Algebras

Let us define the parallel relationships of the notations Q, $Q_{p,q}$, and Q_n; $\mathcal{C}\ell(Q)$, $\mathcal{C}\ell_{p,q}$, and $\mathcal{C}\ell_n$; and $\mathcal{OC}(Q)$, $\mathcal{OC}_{p,q}$, and $\mathcal{OC}_n$. Here $n \in \mathbb{Z}[1,\infty]$ and $p, q \in \mathbb{Z}[0,\infty]$ with $p \geq 1$ or $q \geq 1$.[3] First we need the notions of a general *quadratic form* Q and its associated *symmetric bilinear form* B_Q.[4]

Definition 1.1 (Fauser [7], p. 3)
A quadratic form *on a vector space V^n over $\mathbb{R}$ is a map $Q\colon V^n \to \mathbb{R}$ such that*

$$Q(\alpha x) = \alpha^2 Q(x) \text{ for all } \alpha \in \mathbb{R} \text{ and } x \in V, \text{ and} \tag{1.1a}$$

$$B_Q(x,y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y)) \text{ for all } x, y \in V^n, \tag{1.1b}$$

where $B_Q\colon V^n \times V^n \to \mathbb{R}$ is the symmetric bilinear form *associated with Q by the* polarization relation *given by eq. (1.1b).*

A quadratic form on V^n such that $Q(x) \neq 0$ for all $x \in V^n$ is said to be *nondegenerate*. Let Q be a nondegenerate quadratic form on V^n. If there exists an indexed set of mutually orthogonal[5] vectors $\{\, \mathbf{e}_1, \ldots, \mathbf{e}_p, \mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+q} \,\}$ for

[3] Here we have used the notation $\mathbb{Z}[a,b] \equiv \{\, i \mid i \in \mathbb{Z} \text{ and } a \leq i \leq b \,\}$ introduced in the Preface.

[4] A map with two arguments such that $B\colon U \times V \to W$, where U, V, are W are vector spaces over $\mathbb{R}$, is said to be *bilinear* iff it is linear in both of its arguments. That is, $B(x+y,z) = B(x,z) + B(y,z)$, $B(x,y+z) = B(x,y) + B(x,z)$, and $B(\alpha x, \beta y) = \alpha\beta\, B(x,y)$ for all $\alpha, \beta \in \mathbb{R}$. The *form* part of its name means that for B_Q we have $W = \mathbb{R}$ in the definition of bilinearity just given. Also the word *symmetric* implies that $U = V$, since it means that $B_Q(x,y) = B_Q(y,x)$.

[5] That is, $B_Q(\mathbf{e}_i, \mathbf{e}_j) = 0$, if $i \neq j$. Note also that these vectors are not necessarily normalized to unit length.

V^n such that for all $\mathbf{e}_i$

$$Q(\mathbf{e}_i)\begin{cases} > 0, & \text{for} \quad\quad 1 \le i \le p, \quad \text{and} \\ < 0, & \text{for } p+1 \le i \le p+q = n, \end{cases}$$

we say that Q is of *signature* (p,q)[6] and we may write $Q_{p,q}$ to signify this. If $q = 0$, we then have $p = n$; whereupon we say Q is of *positive signature* n and we may write Q_n to indicate it. Also if $p = 0$, we then have $q = n$; whereupon we say Q is of *negative signature* n and we may write $Q_{0,n}$ to indicate that.

We may also represent that a nondegenerate quadratic form $Q_{p,q}$, Q_n, or $Q_{0,n}$ exits for the vector space V^n by writing $V^{p,q}$, $V^{n,0}$, or $V^{0,n}$, respectively.[7] We symbolize the corresponding Clifford algebras by $\mathcal{C}\ell_{p,q}$, $\mathcal{C}\ell_n$, and $\mathcal{C}\ell_{0,n}$; and the corresponding orientation congruent algebras by $\mathcal{OC}_{p,q}$, $\mathcal{OC}_n$, and $\mathcal{OC}_{0,n}$.[8] When discussing the Clifford or orientation congruent algebra of a general quadratic form Q, or when the signature (p,q) of Q is understood from context, we may also write $\mathcal{C}\ell(Q)$ or $\mathcal{OC}(Q)$, respectively. And when referring to the nondegenerate quadratic form of signature (p,q) associated with the Clifford algebra $\mathcal{C}\ell_{p,q}$, we will usually write simply Q instead of $Q_{p,q}$.

1.2 GR Axioms for the Clifford Algebra $\mathcal{C}\ell_{p,q}$ of a Nondegenerate Quadratic Form

Before we give a set of axioms for $\mathcal{OC}_{p,q}$ we first introduce a compact axiomatic definition of $\mathcal{C}\ell_{p,q}$ adapted from Lounesto's presentation.[9] Then we will expand this compact definition into a longer list of 15 axioms in three sets. Finally, after modifying this axiomatic formulation for $\mathcal{C}\ell_{p,q}$, we obtain a system of 17 axioms for $\mathcal{OC}_{p,q}$.

Hereafter the term *multivector* shall refer to any element of the Clifford algebra $\mathcal{C}\ell_{p,q}$ (or the orientation congruent algebra $\mathcal{OC}_{p,q}$) including those containing a scalar or vector component. Also the Clifford algebra product shall be denoted by an open dot $\circ$.[10]

Definition 1.2 (by Generators and Relations[11])
An associative algebra over $\mathbb{R}$ with unit 1 *is the* Clifford algebra $\mathcal{C}\ell_{p,q}$ *of a nondegenerate quadratic form Q on V^n (with the Clifford product symbolized by*

[6]This (p,q) is, of course, the physicist's signature, not $s = p - q$, the mathematician's version.

[7]Note that we do not use V^n as a brief form of V^nn,0, as we do for the corresponding notations for the Clifford and orientation congruent algebras, because we reserve V^n to indicate the vector space of dimension n that does not necessarily have a quadratic form associated with it.

[8]The degenerate *algebras* $\mathcal{C}\ell_{0,0}$ and $\mathcal{OC}_{0,0}$ also exist, but are not associated with a quadratic form since they are isomorphic with $\mathbb{R}$.

[9]See Ref. [10], pp. 190–2. Chapters 14, 21, and 22 of Ref. [10] also give several other definitions of a Clifford algebra.

[10]Usually Clifford multiplication is indicated by juxtaposition but here we prefer to distinguish between it and orientation congruent multiplication by giving each its own symbol: an open dot $\circ$, and a circled open dot $\circledcirc$, respectively.

an open dot $\circ$*) if it contains* V^n *and* $\mathbb{R} = \mathbb{R} \cdot 1$ *as distinct subspaces so that*

(1) $x \circ x = Q(x)$ *for all* $x \in V^n$*,*
(2) V^n *generates* $\mathcal{C}\ell_{p,q}$ *as an algebra over* $\mathbb{R}$*, and*
(3) $\mathcal{C}\ell_{p,q}$ *is not generated by any proper subset of* V^n*.*

As Lounesto remarks condition (3) of Def. 1.2 ensures that $\mathcal{C}\ell_{p,q}$ so defined is a *universal object* in the category theoretic sense and that the dimension of $\mathcal{C}\ell_{p,q}$ is 2^n. Roughly stated, the universality of an object means that it is unique up to isomorphism under a change of basis and that it is of the maximum size allowed by its definition.[12] Applied works commonly use a long set of axioms similar to those we give next to define the Clifford algebra $\mathcal{C}\ell_{p,q}$; however, usually these works do not also mention the mathematically sophisticated refinement of condition (3).

For reference and completeness we will now spell out Lounesto's axiomatic definition for the Clifford algebra $\mathcal{C}\ell_{p,q}$ of a nondegenerate quadratic form in a longer list of three sets of five axioms.[13] This list starts with two sets of five axioms which are the standard vector space axioms; however, now the vector space contains the multivectors in $\mathcal{C}\ell_{p,q}$ rather than just the vectors in the base space V^n. The first set of axioms gives the properties of multivector addition; the second set, the properties of two-sided scalar multiplication. To these first ten we add the *algebraic* axioms for Clifford multiplication of multivectors given in Axiom Set III.

Axiom III.2 below assumes that $\mathbb{R} \subseteq \mathcal{C}\ell_{p,q}$; that is, that scalars are multivectors. Similarly, Axiom III.5 assumes that $V^n \subseteq \mathcal{C}\ell_{p,q}$; that is, that vectors are multivectors. However, in a more careful interpretation, one says that $\mathbb{R}$ and V^n are present in $\mathcal{C}\ell_{p,q}$ only as isomorphic images. The approach adopted here of identifying $\mathbb{R}$ and V^n with their images in $\mathcal{C}\ell_{p,q}$ creates redundancies in our axiom system that are discussed in detail in the footnotes. There we see that most of the axioms in the second set are subsumed and mirrored in those of the third set; scalar multiplication of multivectors will have become, after all, just Clifford multiplication by a scalar, and so must be consistent with it.

The first axiom set spells out that the set of *multivectors,* $\mathcal{C}\ell_{p,q}$, is an *abelian group* under the operation of *multivector addition.* The *group operation* is symbolized by the *addition sign* $+$.

[11] After Lounesto [10], p. 190.

[12] For a more detailed discussion of universality under the name *unique factorization property,* and in the context of the tensor product of vector spaces, see Shaw [15], pp. 274–7. See also Perwass's thesis [12], p. 18.

[13] These axioms for $\mathcal{C}\ell_{p,q}$ were adapted from those of Perwass [13], pp. 22–24. Shaw ([14], pp. 6,9) was also consulted for the vector space properties postulated in Axiom Set I. But note that this axiom system must be supplemented with conditions (2) and (3), and the requirement that $\mathbb{R}$ and V^n are distinct subspaces, all from Def. 1.2 due to Lounesto.

Axiom Set I Addition of Multivectors (in the Vector Space $\mathcal{C}\ell_{p,q}$)
There exists a binary operation called multivector addition, *symbolized by the* addition sign $+$, *and that is said to produce the* sum *of two multivectors, such that for all* $A, B, C \in \mathcal{C}\ell_{p,q}$

I.1.	$A + B \in \mathcal{C}\ell_{p,q}$,	*Existence and closure of sum*
I.2.	$A + B = B + A$,	*Commutativity*
I.3.	$(A + B) + C = A + (B + C)$,	*Associativity*
I.4.	$A + 0 = A$, *and*	*Existence of an identity*
I.5.	$A + (-A) = 0$.	*Existence of an inverse*[14]

In the second axiom set and below the elements of $\mathbb{R}$ are called *scalars*.

Axiom Set II Two-Sided Scalar Multiplication of Multivectors (in the Vector Space $\mathcal{C}\ell_{p,q}$)
There exists a binary operation called scalar multiplication, *and symbolized by* juxtaposition, *such that for all* $A, B \in \mathcal{C}\ell_{p,q}$*,* $a \in V^n$*, and* $\alpha, \beta \in \mathbb{R}$

II.1.	$\alpha A, A\alpha \in \mathcal{C}\ell_{p,q}$,	*Existence and closure of scalar product*
II.2.	$A\alpha = \alpha A$,	*Commutativity*[15]
II.3.	$(\alpha\beta)A = \alpha(\beta A)$,	*Associativity of left scalar multiplication*
II.4.	$1A = A$,	*Existence of a left identity*
II.5a.	$(\alpha + \beta)A = \alpha A + \beta A$, *and*	*Distributivity of lsm*[16] *over scalar addition*
II.5b.	$\alpha(A + B) = \alpha A + \alpha B$.	*Distributivity of lsm over mv. addition*

Definition 1.3
An algebra *over* $\mathbb{R}$ *is a vector space* W *together with a bilinear*[17] *binary operation,* m*, called the algebra's* product *or* multiplication, *such that* $m\colon W \times W \to W$ *as* $m\colon (x, y) \mapsto m(x, y)$*. Sometimes* $m(x, y)$ *is written as* $x \circledm y$*, where* $\circledm$ *is usually some more abstract symbol such as* $\circ$*, or, simply by juxtaposing the arguments, as* xy*.*

Adding the third set of axioms turns the vector space $\mathcal{C}\ell_{p,q}$ into an associative algebra and relates the quadratic form Q associated with V^n to the Clifford square of the vectors in $\mathcal{C}\ell_{p,q}$. This algebra inherits a unit from the vector space by Axioms II.4 and III.2.

[14] This axiom is derivable from others in the two sets of vector space axioms and the field properties of $\mathbb{R}$ if we *define* $-A$ to be the result of the scalar multiplication $(-1)A$.

[15] Since $\mathbb{R}$ is a field, and thus has a *commutative* multiplication, it is not necessary to assume the existence of right scalar multiplication $A\alpha$ in Axiom II.1. Axiom II.2 may then be taken as a definition of right scalar multiplication as $A\alpha \equiv \alpha A$. See Shaw [14], p. 9, Rem. (b).

[16] We use "lsm" as short for "left scalar multiplication."

[17] A binary operation is bilinear iff it is linear in both of its arguments. Bilinearity implies distributivity of the product over vector space addition. Nevertheless, we explicitly include the distributive property in the axioms. For a more general defintion of bilinearity see fn. 4.

Axioms III.4 and III.5 have been placed at the end of the list because these two of the fifteen will be substantially modified for the orientation congruent algebra $\mathcal{OC}_{p,q}$.

Axiom Set III Clifford Multiplication of Multivectors (in the Algebra $\mathcal{C}\ell_{p,q}$)
There exists an algebraic product called Clifford multiplication, *and symbolized by an* open dot $\circ$, *such that for all* $A, B, C \in \mathcal{C}\ell_{p,q}$, $a \in V^n$, *and* $\alpha \in \mathbb{R}$

III.1.	$A \circ B \in \mathcal{C}\ell_{p,q}$,	*Existence and closure of product*
III.2.	$\alpha \circ A = \alpha A, \quad A \circ \alpha = A\alpha$,	*Equality with l. & r. scalar mult.*[18]
III.3a.	$A \circ (B + C) = A \circ B + A \circ C$,	*Left distributivity over mv. add.*
III.3b.	$(B + C) \circ A = B \circ A + C \circ A$,	*Rt. distributivity over mv. add.*[19]
III.4.	$(A \circ B) \circ C = A \circ (B \circ C)$, *and*	*Associativity*[20]
III.5.	$a^2 \equiv a \circ a = Q(a)$.	*Equality of the square and quadratic form of vectors*

Implicit in the expressions of Axiom Set III is the usual parentheses-sparing convention of performing Clifford multiplications before performing multivector additions. Specifically, in Axioms III.3a and III.3b this operator precedence rule is applied on the right sides of the equations.

1.3 GR Axioms for the Orientation Congruent Algebra $\mathcal{OC}_{p,q}$ of a Nondegenerate Quadratic Form

We consider now another list of 15 axioms parallel to the one above, but modified. Then, we add two new axioms to obtain a list of 17 axioms[21] that will provide the axiomatic foundation for the $\mathcal{OC}_{p,q}$ algebra.

The first ten axioms in Axiom Sets I and II and the first three axioms in Axiom Set III are changed, but only trivially with the replacement of the terms and symbols referring to Clifford algebra with those referring to orientation congruent algebra. Therefore, we do not list the first ten of these axioms in their modified forms; however, we do list the first three axioms of Axiom Set III so changed. Next, we briefly describe the material changes and additions to Axiom Set III before making them.

[18] As mentioned above, we have assumed that $\mathbb{R} \subseteq \mathcal{C}\ell(Q)$; that is, that scalars are multivectors. Therefore, the properties of scalar multiplication given in Axiom Set II are partially subsumed under those of Clifford multiplication given in this axiom set. In particular, this axiom and the one above it make Axiom II.1 redundant and it may be dropped.

[19] These Axioms III.3a and III.3b of the distributivity of Clifford multiplication, with the help of Axiom III.2, imply the (now redundant) Axioms II.5a and II.5b of the distributivity of left scalar multiplication.

[20] This Axiom III.4 of the associativity of Clifford multiplication, with the help of Axiom III.2, implies the (now redundant) Axiom II.3 of the associativity of scalar multiplication.

[21] As with that for $\mathcal{C}\ell_{p,q}$ this axiom system for $\mathcal{OC}_{p,q}$ must also be supplemented with suitably modified conditions similar to (2) and (3) of Def. 1.2 and the requirement that $\mathbb{R}$ and V^n are distinct subspaces, again all adapted from Lounesto ([10], p. 190).

The nontrivial changes required to the axioms in Axiom Set III are to

1) restrict the product used in Axiom III.4 for associativity from the orientation congruent product to the outer product,

2) extend the domain of Axiom III.5 for the equality of the algebra product square of a vector and its quadratic form to nonscalar blades,

3) add a new axiom for the existence of a *counit* $\boldsymbol{\omega}_{\mathscr{A}}$ *of a set of multivectors* $\mathscr{A}$, and

4) add a second new axiom for the *generalized commutativity of the right* $\boldsymbol{\omega}_{\mathscr{A}}$*-complement* that supplements the restricted Axiom III.4.

All numbers of the modified axioms for the orientation congruent algebra will be marked with primes to indicated their correspondence with the original axioms for the Clifford algebra. For consistency the numbers of the new sixteenth and seventeenth axioms will also be primed.

Axiom Set III′ Orientation Congruent Multiplication of Multivectors (in the Algebra $\mathcal{OC}_{p,q}$)
There exists an algebraic product called orientation congruent multiplication, *and symbolized by an* circled open dot $\circledcirc$, *such that for all* $A, B, C \in \mathcal{OC}_{p,q}$ *and* $\alpha \in \mathbb{R}$

III.1′.	$A \circledcirc B \in \mathcal{OC}_{p,q}$,	*Existence and closure of product*
III.2′.	$\alpha \circledcirc A = \alpha A, \quad A \circledcirc \alpha = A\alpha$,	*Equality with l. & r. scalar mult.*
III.3a′.	$A \circledcirc (B + C) = A \circledcirc B + A \circledcirc C$,	*Left distributivity over mv. add.*
III.3b′.	$(B + C) \circledcirc A = B \circledcirc A + C \circledcirc A$.	*Rt. distributivity over mv. add.*

Before presenting the next axiom we pause to make some definitions which will also be used later.

Definition 1.4

a) A multivector $A \in \mathcal{OC}_{p,q}$ *is called an* r-blade *iff, for some* $r \in \mathbb{Z}[2, n]$, *it can be written as an orientation congruent multiproduct, with any grouping into binary products, of* r *mutually anticommuting vectors. That is,* $A = \boldsymbol{a}_1 \circledcirc \cdots \circledcirc \boldsymbol{a}_i \circledcirc \cdots \circledcirc \boldsymbol{a}_r$ *where all* $\boldsymbol{a}_i \in V^n$ *and* $\boldsymbol{a}_i \circledcirc \boldsymbol{a}_j = -\boldsymbol{a}_j \circledcirc \boldsymbol{a}_i$ *for all* $i \neq j$.

b) We also define the term 1-blade *to mean* vector, *and the term* 0-blade *to mean* scalar. *And we interpret the multiproduct notation* $A = \boldsymbol{a}_1 \circledcirc \cdots \circledcirc \boldsymbol{a}_i \circledcirc \cdots \circledcirc \boldsymbol{a}_r$ *to be the vector* $A = \boldsymbol{a}_1$, *when* $r = 1$, *and some scalar* $A = \alpha$ *when* $r = 0$.

c) All zero-valued r*-blades are considered to be equivalent for any* $r \in \mathbb{Z}[0, n]$. *Thus,* 0 *represents a blade of indeterminate grade.*

d) An r-vector *is defined as a linear combination of* r*-blades.*

Definition 1.5

a) The outer product *of A_r and B_s, written with a* wedge $\wedge$, *is defined for any r-vector and s-vector $A_r, B_s \in \mathcal{OC}_{p,q}$ as the $(r+s)$-grade part of their orientation congruent product*

$$A_r \wedge B_s \equiv \langle A_r \circledcirc B_s \rangle_{r+s}. \tag{1.2}$$

b) The outer product of general multivectors $A, B \in \mathcal{OC}_{p,q}$ is then defined by

$$A \wedge B \equiv \sum_{r,s} \langle A \rangle_r \wedge \langle B \rangle_s = \sum_r \langle A \rangle_r \wedge B = \sum_s A \wedge \langle B \rangle_s. \tag{1.3}$$

Now we may continue with the next axiom.

Axiom Set III′ Orientation Congruent Multiplication (continued)
And such that for all $A, B, C \in \mathcal{OC}_{p,q}$

III.4′. $(A \wedge B) \wedge C = A \wedge (B \wedge C)$. *Associativity of outer product*

And such that for any $A \in \mathcal{OC}_{p,q}$ such that A is the nonscalar r-blade $A = \boldsymbol{a}_1 \circledcirc \cdots \circledcirc \boldsymbol{a}_i \circledcirc \cdots \circledcirc \boldsymbol{a}_r$

III.5′. $A^2 \equiv A \circledcirc A = Q(\boldsymbol{a}_1) \cdots Q(\boldsymbol{a}_i) \cdots Q(\boldsymbol{a}_r)$. *Equality of the square of an r-blade and the product of the quadratic forms of its vectors*

In the next axiom (and the sequel) we use the adjective *unitary* to refer to *equality to* ± 1; elsewhere we use the phrase *the unitaries* to refer to 1 *and* -1.

Axiom Set III′ Orientation Congruent Multiplication (continued)
And such that for all nonempty sets of multivectors $\varnothing \subset \mathscr{A} \subseteq \mathcal{OC}_{p,q}$ there exists a (nonunique) blade called a counit[22] *of $\mathscr{A}$ and symbolized by a subscripted, boldface, lower case omega as $\boldsymbol{\omega}_{\mathscr{A}}$, such that $\boldsymbol{\omega}_{\mathscr{A}} \in \mathcal{OC}_{p,q}$ and for all $A \in \mathscr{A}$*

III.6a′. $\boldsymbol{\omega}_{\mathscr{A}} \neq \pm 1$, *Nonunitary*

III.6b′. ${\boldsymbol{\omega}_{\mathscr{A}}}^2 \equiv \boldsymbol{\omega}_{\mathscr{A}} \circledcirc \boldsymbol{\omega}_{\mathscr{A}} = \pm 1$, *and* *Unitary $\mathcal{OC}$ square*

III.6c′. $A \circledcirc \boldsymbol{\omega}_{\mathscr{A}} = \boldsymbol{\omega}_{\mathscr{A}} \circledcirc A$. *$\mathscr{A}$-universal commutativity*

Applying the last axiom we make the following definitions.

[22] The name "counit" is a contraction of the phrase "coscalar unit." The "unit" part of the name is appropriate because a counit behaves algebraically like the unit. Indeed, for the set $\mathscr{A} = \mathcal{OC}_{p,q}$, the unitaries, 1 and -1, are the only elements other than $\boldsymbol{\Omega}$ and $-\boldsymbol{\Omega}$ (see the next Def. 1.6) that have properties (b) and (c) of Axiom III.6′. And the "co" part of the name is consistent with the definition of a *coscalar* as an element of $\mathcal{OC}_{p,q}$ that has a *complementary grade* or *cograde* of $0 = n - k$ because it also has a grade of $k = n$ in the set of multivectors $\mathcal{OC}_{p,q}$ with $n = p + q$. Generally, when working in the algebra $\mathcal{OC}_{p,q}$, a minimal grade counit $\boldsymbol{\omega}_{\mathscr{A}}$ of a nonempty set of multivectors $\mathscr{A}$ has a cograde of $0 = m - k$ (or a grade of $k = m$) *relative to* the smallest odd $m = r + s$ such that $\mathscr{A} \subseteq \mathcal{OC}_{r,s} \subseteq \mathcal{OC}_{p,q}$.

Definition 1.6

a) If $\mathscr{A} = \mathcal{OC}_{p,q}$ we call any $\boldsymbol{\omega}_{\mathscr{A}}$ "a" counit of the algebra $\mathcal{OC}_{p,q}$ *and we usually write such an $\boldsymbol{\omega}_{\mathscr{A}}$ using a boldface uppercase omega as $\boldsymbol{\Omega}$.*

b) In fact, if $n = p + q$ is even there are no counits of "$\mathcal{OC}_{p,q}$" and we have only an imperfect *orientation congruent algebra (of a nondegenerate quadratic form) that does not satisfy the last two Axioms III.6′ and III.7′ of Axiom Set III′. We symbolize an imperfect orientation congruent algebra as $\mathcal{IOC}_{p,q}$.*[23]

c) But, if n is odd there are exactly two counits of the algebra that differ only by sign. These are $\pm\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_i \wedge \cdots \wedge \mathbf{e}_n$ for $\mathbf{e}_i \in \mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, where $\mathscr{B}$ is an ordered, orthonormal set of basis vectors for V^n. Choosing one of these counits establishes an orientation *for $\mathcal{OC}_{p,q}$. An $\boldsymbol{\Omega}$ so chosen will be called* "the" counit of the algebra $\mathcal{OC}_{p,q}$, *or, simply,* the couint. *We may write it with an underline as $\underline{\boldsymbol{\Omega}}$. If an ordered, orthonormal set of basis vectors for V^n has been specified and no choice for the couint has been explicitly made, we assume that the counit is the one which is the product of the basis vectors in basis order $\underline{\boldsymbol{\Omega}} = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n$.*[24]

In preparation for the next and final axiom of Axiom Set III′ we define the following notions and notations.

Definition 1.7

We use a superscript $\boldsymbol{\omega}_{\mathscr{A}}$ attached to a multivector $A \in \mathcal{OC}_{p,q}$ to mean

$$A^{\boldsymbol{\omega}_{\mathscr{A}}} \equiv A \circledcirc \boldsymbol{\omega}_{\mathscr{A}}. \tag{1.4}$$

And, similarly, for left-sided multiplication by $\boldsymbol{\omega}_{\mathscr{A}}$ we define

$${}^{\boldsymbol{\omega}_{\mathscr{A}}}A \equiv \boldsymbol{\omega}_{\mathscr{A}} \circledcirc A. \tag{1.5}$$

We call these operations right (left) $\boldsymbol{\omega}_{\mathscr{A}}$-complementation, *or* counit complementation by $\boldsymbol{\omega}_{\mathscr{A}}$, *and we give them precedence over orientation congruent, Clifford, and outer product multiplications.*

Axiom Set III′ Orientation Congruent Multiplication (finished)

And such that for all $\mathscr{A}$ that are nonempty sets of multivectors, $\varnothing \subset \mathscr{A} \subseteq \mathcal{OC}_{p,q}$, all couints $\boldsymbol{\omega}_{\mathscr{A}}$ of $\mathscr{A}$, and all $A, B \in \mathscr{A}$

III.7′. $A^{\boldsymbol{\omega}_{\mathscr{A}}} \circledcirc B = A \circledcirc B^{\boldsymbol{\omega}_{\mathscr{A}}} = (A \circledcirc B)^{\boldsymbol{\omega}_{\mathscr{A}}}$.	*Generalized commutativity of right $\boldsymbol{\omega}_{\mathscr{A}}$-complementation*

Precisely now with the presentation of this final axiom in Axiom Set III′ we have completed the construction a GR axiom system for the orientation congruent algebra of a nondegenerate quadratic form.

[23] An $\mathcal{IOC}_{r,s}$ with $m = r + s$ can always be extended to an $\mathcal{OC}_{p,q}$ with $n = m + 1 = p + q$ and having primed basis vectors by adding another basis vector $\mathbf{e}_{m+1}{}' = \mathbf{e}_n{}'$ (making $p = r$ and $q = s+1$) or $\mathbf{e}_{r+1}{}' = \mathbf{e}_p{}'$ (making $p = r+1$ and $q = s$) in a signature-ordered, orthogonal set of basis vectors.

[24] In this case the counit $\underline{\boldsymbol{\Omega}}$ is the same element in $\mathcal{OC}_{p,q}$ as what is called, in the language of *geometric algebra* (Clifford algebra given a geometric interpretation), the *unit pseudoscalar* I associated with a orthonormal *frame* (set of basis vectors) for $\mathcal{C}\ell_{p,q}$. Also the q part of the signature (p, q) of the quadratic form of $\mathcal{OC}_{p,q}$ determines the sign of the orientation congruent square of a counit of the algebra by $\boldsymbol{\Omega}^2 = (-\boldsymbol{\Omega})^2 = (-1)^q$.

Remarks 1.1

a) The last two Axioms III.6′ and III.7′ together with Axiom III.4′ replace Axiom III.4 expressing the associativity of the Clifford product. Associativity is just one member of the class of possible "bracket shifting rules."

b) Axiom III.4′ partially replaces the general associativity of the Clifford product with that of the outer product. The outer product is derived by a grade selection from the orientation congruent product. Equivalently, this axiom may also be viewed as restricting the domain of applicability of the orientation congruent product to two blades whose component vectors mutually anticommute when combined as one group. This axiom has a direct analog as a theorem in all Clifford algebras $\mathcal{C}\ell_{p,q}$.

c) Axioms III.6′ and III.7′ supplement Axiom III.4′ with a pair of commutative and bracket shifting rules both involving $\boldsymbol{\omega}_{\mathscr{A}}$, and both more complicated, but generally applicable. These two axioms have direct analogs as theorems in all Clifford algebras $\mathcal{C}\ell_{p,q}$ with odd $n = p + q$.

d) In summary, we might say that to transform the axioms for $\mathcal{C}\ell_{p,q}$ into those for $\mathcal{OC}_{p,q}$ we have traded an expansion of the domain of applicability of Axiom III.5 from vectors to blades in Axiom III.5′ for a restriction of the domain of applicability of Axiom III.4 with its consequent fragmentation into the three Axioms III.4′, III.6′, and III.7′.

1.4 Other Axiom Systems

This subsection discusses some alternative axiomatic formulations of the orientation congruent algebra and previews the next section.

The literature provides other axiomatic formulations of Clifford algebras of varying generality; we will consider their adaptability to the orientation congruent algebra. These other Clifford algebra axiom systems range, for example, from those describing a Clifford algebra as an ideal of a tensor algebra ([10], pp. 193f), or describing it in category-theoretic terms as the universal object of a quadratic algebra ([10], pp. 192f), or embedding it as a subalgebra of the associated exterior algebra's endomorphism algebra through the Chevalley-operator representation (which Chevalley [5] based on the Cartan decomposition formula),[25] or describing it as a Hopf gebra[26] using tensor algebra and category theory expressed in commutative and tangle diagrams ([8], chs. 3– 5), to providing a multiplication rule for basis blades represented by n-tuples of binary digits called multi-indices ([10], ch. 21).[27]

Only three of these approaches to the axiomatization of Clifford algebra are directly convertible to the orientation congruent algebra. One is the definition

[25] This decomposition formula is credited to E. Cartan by Crumeyrolle ([6], p. 44) and Abłamowicz ([2] p. 463). Chevalley's method is also used by Lounesto ([10], ch. 22), Crumeyrolle ([6], p. 45), and Oziewicz [11]. It is also implicit in the paper of Fernández, Moya, and Rodrigues ([9], p. 15).

[26] This is not a misprint. Without going into details, a Hopf gebra is a more general structure than a Hopf algebra ([8], p. 65).

[27] This last is really a specialized form of GR axiomitization.

as a universal object of quadratic algebras. The modification required is simply using nonassociative quadratic algebras in place of the (assumed) associative quadratic algebras and adding other relations to represent Axioms III.4′, III.5′, III.6′, and III.7′. However, since this very abstract definition is nonconstructive, it is not useful for calculating the orientation congruent product.

It is only the last two definitions, one based on Hopf gebra and the other on a multiplication rule for basis blades that are both adaptable and useful. That is because the other approaches are based on intrinsically associative algebras. Hopf gebras, however, are not ruled out; associativity is not necessary for their definition ([8], p. 65). Also as demonstrated by Fauser [8] the Hopf gebraic approach is very fruitful in producing grade-free computational algorithms for very general forms of Clifford algebras.

The last definition from a multiplication rule for basis blades is easily generalizable to Clifford-like algebras. These are essentially the algebras of the Clifford product but as modified by a sign rule that may differ from the standard Clifford algebra one ([10], pp. 284ff). The Clifford-like algebras, however, are not necessarily associative. They may also have other properties that vary from those of Clifford algebra. In the next section we will construct the explicitly Clifford-like *sigma orientation congruent algebra* $\sigma\mathcal{OC}_{p,q}$ which uses a multiplication that is the Clifford product times the *sign factor function* σ.

In the next section we also prove the deductive equivalence of the set of primed axioms for the orientation congruent algebra $\mathcal{OC}_{p,q}$ with that of the unprimed axioms for the Clifford algebra $\mathcal{C}\ell_{p,q}$ supplemented by an existence axiom for the sigma orientation congruent product. In so doing we establish that the sigma orientation congruent algebra of a nondegenerate quadratic form is isomorphic to the corresponding orientation congruent algebra. Then, instead of reasoning directly from the axioms of the current section, we can also prove theorems for the orientation congruent algebra by interpreting its product as the sigma orientation congruent product and manipulating ordinary algebraic expressions derived from the sign factor function while citing verified Clifford algebra theorems.

Actually, in the rest of this paper the sigma form of the orientation congruent product will be the basis for investigating the $\mathcal{OC}_{p,q}$ algebra. Indeed, while simply proving the equivalence of the orientation congruent product and the sigma orientation congruent product in the next section other proofs of some assertions made in this section will naturally fall out as byproducts. One statement with such an incidental proof is that the orientation congruent algebra $\mathcal{OC}_{p,q}$ exists only for base spaces V^n of odd dimension, and, complementarily, that the imperfect orientation congruent algebra $\mathcal{IOC}_{p,q}$ exists only for base spaces V^n of even dimension.

1.5 Multiplication Tables

We end this section with the multiplication tables for the Clifford algebra $\mathcal{C}\ell_3$ (Tab. 1.1), and the orientation congruent algebras $\mathcal{OC}_3$ (Tab. 1.2) and $\mathcal{OC}_5$ (Tab. 1.3). In these tables the basis blades are written with multi-indices so that, for example, $\mathbf{e}_{23} = \mathbf{e}_2 \circ \mathbf{e}_3$ and $\mathbf{e}_{23} = \mathbf{e}_2 \circledcirc \mathbf{e}_3$. Also the counits of the orientation congruent algebras are written as $\mathbf{\Omega} = \mathbf{e}_{123} = \mathbf{e}_1 \circledcirc \mathbf{e}_2 \circledcirc \mathbf{e}_3$ and $\mathbf{\Omega} = \mathbf{e}_{12345} = \mathbf{e}_1 \circledcirc \mathbf{e}_2 \circledcirc \mathbf{e}_3 \circledcirc \mathbf{e}_4 \circledcirc \mathbf{e}_5$.[28] We have also used this same symbol $\mathbf{\Omega}$ for the pseudoscalar $\mathbf{e}_{123} = \mathbf{e}_1 \circ \mathbf{e}_2 \circ \mathbf{e}_3$ of the Clifford algebra $\mathcal{C}\ell_3$.

The underlined entries in the orientation congruent algebra multiplication tables are oppositely signed compared to those in the tables for the corresponding Clifford algebras. Also the entries in red highlighted cells in all tables are negatively signed.

In the orientation congruent algebra multiplication tables (Tabs. 1.2 and 1.3) the red highlighting makes the reflection symmetry of the negative signs about the central horizontal and vertical lines easy to see. In the multiplication table for the Clifford algebra $\mathcal{C}\ell_3$ (Tab. 1.1) the negative signs have no obvious symmetry. Both the symmetry in Tabs. 1.2 and 1.3 and its lack in Tab. 1.1 result from displaying these tables in a canonical form specific to the orientation congruent algebra.

In all three of these tables the factor basis blades in the leftmost column and in the top row are in *graded, reflected complementary order.* In addition, an ordering for the multi-indices of the basis blades has been chosen so that the set of basis blades is *coherently oriented.* These two requirements define a *canonical* arrangement for the orientation congruent algebra multiplication tables (but not for the Clifford algebra ones).

Similarly, we can define a canonical arrangement of the Clifford algebra multiplication tables comprising *Gray code order* ([10], pp. 281ff) for the factor basis blades and *increasing numerical order* within the sequence of multi-indices for each basis blade. Clifford algebra multiplication tables in this form display their negatively signed entries symmetrically, just as do the orientation congruent algebra multiplication tables in their canonical form. However, this symmetry is of a different kind than that of the orientation congruent algebra multiplication tables. These remarks on the symmetry of canonically arranged multiplication tables for the Clifford and orientation congruent algebras will be expanded in a later section of this paper.

[28] Because it conflicts with another usage in these tables we have forgone the underlining of omegas to symbolize the counits of these $\mathcal{OC}$ algebras (the convention established in Def. 1.6 of the previous section).

$a \circledcirc b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_1$	$\boldsymbol{\Omega}$	$\mathbf{e}_3$	$\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\boldsymbol{\Omega}$	$\mathbf{e}_1$	$-\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\boldsymbol{\Omega}$	-1	$\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_3$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$\boldsymbol{\Omega}$	$-\mathbf{e}_1$	$-\mathbf{e}_{23}$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_2$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	-1	$-\mathbf{e}_1$
$\boldsymbol{\Omega}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	-1

Table 1.1: The Multiplication Table for the Clifford Algebra $\mathcal{C}\ell_3$. The factors are in graded, reflected complementary order. Their indices are ordered so that the the basis blades have coherent orientations. Red cells contain negative entries.

$a \circledcirc b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\underline{-\mathbf{e}_2}$	$\underline{\mathbf{e}_3}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$\underline{\mathbf{e}_1}$	$\boldsymbol{\Omega}$	$\underline{-\mathbf{e}_3}$	$\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\boldsymbol{\Omega}$	$\underline{-\mathbf{e}_1}$	$\underline{\mathbf{e}_2}$	$\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\underline{\mathbf{e}_2}$	$\underline{-\mathbf{e}_1}$	$\boldsymbol{\Omega}$	$\underline{1}$	$\underline{-\mathbf{e}_{23}}$	$\underline{\mathbf{e}_{31}}$	$\underline{\mathbf{e}_3}$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\underline{-\mathbf{e}_3}$	$\boldsymbol{\Omega}$	$\underline{\mathbf{e}_1}$	$\underline{\mathbf{e}_{23}}$	$\underline{1}$	$\underline{-\mathbf{e}_{12}}$	$\underline{\mathbf{e}_2}$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$	$\underline{\mathbf{e}_3}$	$\underline{-\mathbf{e}_2}$	$\underline{-\mathbf{e}_{31}}$	$\underline{\mathbf{e}_{12}}$	$\underline{1}$	$\underline{\mathbf{e}_1}$
$\boldsymbol{\Omega}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\underline{\mathbf{e}_3}$	$\underline{\mathbf{e}_2}$	$\underline{\mathbf{e}_1}$	$\underline{1}$

Table 1.2: The Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_3$. The same orderings of factors and their indices are used in this table as for the corresponding Clifford algebra $\mathcal{C}\ell_3$. Red cells contain negative entries. The underlined entries are oppositely signed compared to those in Tab. 1.1.

Table 1.3: The Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_5$

$a \circledcirc b$	1	e_{1}	e_{2}	e_{3}	e_{4}	e_{5}	e_{34}	e_{25}	e_{45}	e_{31}	e_{51}	e_{42}	e_{12}	e_{53}	e_{23}	e_{14}	e_{235}	e_{145}	e_{124}	e_{345}	e_{135}	e_{234}	e_{245}	e_{123}	e_{134}	e_{125}	e_{1234}	e_{1253}	e_{1245}	e_{3145}	e_{2345}	Ω
1	1	e_{1}	e_{2}	e_{3}	e_{4}	e_{5}	e_{34}	e_{25}	e_{45}	e_{31}	e_{51}	e_{42}	e_{12}	e_{53}	e_{23}	e_{14}	e_{235}	e_{145}	e_{124}	e_{345}	e_{135}	e_{234}	e_{245}	e_{123}	e_{134}	e_{125}	e_{1234}	e_{1253}	e_{1245}	e_{3145}	e_{2345}	Ω
e_{1}	e_{1}	1	e_{12}	$-e_{31}$	e_{14}	$-e_{51}$	e_{134}	e_{125}	e_{145}	$\underline{e_{3}}$	$\underline{e_{5}}$	$-e_{124}$	$\underline{-e_{2}}$	$-e_{135}$	e_{123}	$\underline{-e_{4}}$	$-e_{1253}$	e_{45}	$-e_{42}$	$-e_{3145}$	$-e_{53}$	e_{1234}	e_{1245}	e_{23}	e_{34}	e_{25}	$\underline{-e_{234}}$	$\underline{e_{235}}$	$\underline{-e_{245}}$	$\underline{e_{345}}$	Ω	e_{2345}
e_{2}	e_{2}	$-e_{12}$	1	e_{23}	$-e_{42}$	e_{25}	e_{234}	$\underline{-e_{5}}$	e_{245}	e_{123}	e_{125}	$\underline{e_{4}}$	$\underline{e_{1}}$	$-e_{235}$	$\underline{-e_{3}}$	$-e_{124}$	$-e_{53}$	$-e_{1245}$	$-e_{14}$	e_{2345}	e_{1253}	e_{34}	e_{45}	e_{31}	$-e_{1234}$	e_{51}	$\underline{e_{134}}$	$\underline{-e_{135}}$	$\underline{e_{145}}$	Ω	$\underline{-e_{345}}$	e_{3145}
e_{3}	e_{3}	e_{31}	$-e_{23}$	1	e_{34}	$-e_{53}$	$\underline{-e_{4}}$	$-e_{235}$	e_{345}	$\underline{-e_{1}}$	e_{135}	e_{234}	e_{123}	$\underline{e_{5}}$	$\underline{e_{2}}$	$-e_{134}$	$-e_{25}$	e_{3145}	e_{1234}	e_{45}	e_{51}	e_{42}	$-e_{2345}$	e_{12}	$-e_{14}$	$-e_{1253}$	$\underline{-e_{124}}$	$\underline{e_{125}}$	Ω	$\underline{-e_{145}}$	$\underline{e_{245}}$	e_{1245}
e_{4}	e_{4}	$-e_{14}$	e_{42}	$-e_{34}$	1	e_{45}	$\underline{e_{3}}$	$-e_{245}$	$\underline{-e_{5}}$	$-e_{134}$	e_{145}	$\underline{-e_{2}}$	e_{124}	e_{345}	e_{234}	$\underline{e_{1}}$	e_{2345}	e_{51}	e_{12}	e_{53}	$-e_{3145}$	e_{23}	$-e_{25}$	$-e_{1234}$	$-e_{31}$	e_{1245}	$\underline{e_{123}}$	Ω	$\underline{-e_{125}}$	$\underline{e_{135}}$	$\underline{-e_{235}}$	e_{1253}
e_{5}	e_{5}	e_{51}	$-e_{25}$	e_{53}	$-e_{45}$	1	e_{345}	$\underline{e_{2}}$	$\underline{e_{4}}$	$-e_{135}$	$\underline{-e_{1}}$	$-e_{245}$	e_{125}	$\underline{-e_{3}}$	e_{235}	e_{145}	e_{23}	e_{14}	$-e_{1245}$	e_{34}	$-e_{31}$	$-e_{2345}$	$-e_{42}$	e_{1253}	e_{3145}	e_{12}	Ω	$\underline{-e_{123}}$	$\underline{e_{124}}$	$\underline{-e_{134}}$	$\underline{e_{234}}$	e_{1234}
e_{34}	e_{34}	e_{134}	e_{234}	$\underline{e_{4}}$	$\underline{-e_{3}}$	e_{345}	$\underline{1}$	e_{2345}	$\underline{e_{53}}$	$\underline{-e_{14}}$	e_{3145}	$\underline{e_{23}}$	e_{1234}	$\underline{-e_{45}}$	$\underline{-e_{42}}$	$\underline{e_{31}}$	$\underline{e_{245}}$	$\underline{-e_{135}}$	$\underline{-e_{123}}$	$\underline{e_{5}}$	$\underline{e_{145}}$	$\underline{e_{2}}$	$\underline{-e_{235}}$	$\underline{e_{124}}$	$\underline{e_{1}}$	Ω	$\underline{e_{12}}$	$\underline{-e_{1245}}$	$\underline{e_{1253}}$	$\underline{e_{51}}$	$\underline{e_{25}}$	$\underline{e_{125}}$
e_{25}	e_{25}	e_{125}	$\underline{e_{5}}$	$-e_{235}$	$-e_{245}$	$\underline{-e_{2}}$	e_{2345}	$\underline{1}$	$\underline{-e_{42}}$	$-e_{1253}$	e_{12}	$\underline{e_{45}}$	$\underline{-e_{51}}$	$\underline{-e_{23}}$	e_{53}	$-e_{1245}$	$\underline{-e_{3}}$	$\underline{e_{124}}$	$\underline{-e_{145}}$	$\underline{-e_{234}}$	$\underline{e_{123}}$	$\underline{e_{345}}$	$\underline{-e_{4}}$	$\underline{-e_{135}}$	Ω	$\underline{e_{1}}$	$\underline{-e_{3145}}$	$\underline{-e_{31}}$	$\underline{-e_{14}}$	e_{1234}	$\underline{e_{34}}$	$\underline{e_{134}}$
e_{45}	e_{45}	e_{145}	e_{245}	e_{345}	$\underline{e_{5}}$	$\underline{-e_{4}}$	$\underline{-e_{53}}$	$\underline{e_{42}}$	$\underline{1}$	e_{3145}	$\underline{e_{14}}$	$\underline{-e_{25}}$	e_{1245}	$\underline{e_{34}}$	e_{2345}	$\underline{-e_{51}}$	$\underline{-e_{234}}$	$\underline{e_{1}}$	$\underline{e_{125}}$	$\underline{e_{3}}$	$\underline{-e_{134}}$	$\underline{e_{235}}$	$\underline{e_{2}}$	Ω	$\underline{e_{135}}$	$\underline{-e_{124}}$	$\underline{-e_{1253}}$	$\underline{e_{1234}}$	$\underline{e_{12}}$	$\underline{e_{31}}$	$\underline{e_{23}}$	$\underline{e_{123}}$
e_{31}	e_{31}	$\underline{-e_{3}}$	e_{123}	$\underline{e_{1}}$	$\underline{-e_{134}}$	$\underline{-e_{135}}$	$\underline{e_{14}}$	$\underline{-e_{1253}}$	e_{3145}	$\underline{1}$	$\underline{-e_{53}}$	$\underline{-e_{1234}}$	$\underline{e_{23}}$	$\underline{e_{51}}$	$\underline{-e_{12}}$	$\underline{-e_{34}}$	$\underline{-e_{125}}$	$\underline{-e_{345}}$	$\underline{e_{234}}$	$\underline{e_{145}}$	$\underline{-e_{5}}$	$\underline{-e_{124}}$	Ω	$\underline{e_{2}}$	$\underline{-e_{4}}$	$\underline{e_{235}}$	$\underline{-e_{42}}$	$\underline{-e_{25}}$	e_{2345}	$\underline{e_{45}}$	$\underline{-e_{1245}}$	$\underline{e_{245}}$
e_{51}	e_{51}	$\underline{-e_{5}}$	e_{125}	e_{135}	e_{145}	$\underline{e_{1}}$	e_{3145}	$\underline{-e_{12}}$	$\underline{-e_{14}}$	$\underline{e_{53}}$	$\underline{1}$	e_{1245}	$\underline{e_{25}}$	$\underline{-e_{31}}$	e_{1253}	$\underline{e_{45}}$	$\underline{e_{123}}$	$\underline{e_{4}}$	$\underline{-e_{245}}$	$\underline{e_{134}}$	$\underline{e_{3}}$	Ω	$\underline{e_{124}}$	$\underline{-e_{235}}$	$\underline{-e_{345}}$	$\underline{e_{2}}$	$\underline{e_{2345}}$	$\underline{e_{23}}$	$\underline{e_{42}}$	$\underline{e_{34}}$	$\underline{-e_{1234}}$	$\underline{e_{234}}$
e_{42}	e_{42}	$\underline{-e_{124}}$	$\underline{-e_{4}}$	e_{234}	$\underline{e_{2}}$	$\underline{-e_{245}}$	$\underline{-e_{23}}$	$\underline{-e_{45}}$	$\underline{e_{25}}$	$\underline{-e_{1234}}$	e_{1245}	$\underline{1}$	$\underline{-e_{14}}$	$-e_{2345}$	$\underline{e_{34}}$	$\underline{e_{12}}$	$\underline{e_{345}}$	$\underline{e_{125}}$	$\underline{-e_{1}}$	$\underline{-e_{235}}$	Ω	$\underline{e_{3}}$	$\underline{-e_{5}}$	$\underline{e_{134}}$	$\underline{-e_{123}}$	$\underline{-e_{145}}$	$\underline{-e_{31}}$	$\underline{e_{3145}}$	$\underline{e_{51}}$	$\underline{-e_{1253}}$	$\underline{-e_{53}}$	$\underline{e_{135}}$
e_{12}	e_{12}	$\underline{e_{2}}$	$\underline{-e_{1}}$	e_{123}	e_{124}	e_{125}	e_{1234}	$\underline{e_{51}}$	e_{1245}	$\underline{-e_{23}}$	$\underline{-e_{25}}$	$\underline{e_{14}}$	$\underline{1}$	e_{1253}	$\underline{e_{31}}$	$\underline{-e_{42}}$	$\underline{-e_{135}}$	$\underline{e_{245}}$	$\underline{e_{4}}$	Ω	$\underline{e_{235}}$	$\underline{-e_{134}}$	$\underline{-e_{145}}$	$\underline{e_{3}}$	$\underline{e_{234}}$	$\underline{e_{5}}$	$\underline{e_{34}}$	$\underline{e_{53}}$	$\underline{e_{45}}$	$\underline{-e_{2345}}$	$\underline{e_{3145}}$	$\underline{e_{345}}$
e_{53}	e_{53}	$\underline{-e_{135}}$	$\underline{-e_{235}}$	$\underline{-e_{5}}$	e_{345}	$\underline{e_{3}}$	$\underline{e_{45}}$	$\underline{e_{23}}$	$\underline{-e_{34}}$	$\underline{-e_{51}}$	$\underline{e_{31}}$	$-e_{2345}$	e_{1253}	$\underline{1}$	$\underline{-e_{25}}$	$-e_{3145}$	$\underline{-e_{2}}$	$\underline{-e_{134}}$	Ω	$\underline{e_{4}}$	$\underline{-e_{1}}$	$\underline{e_{245}}$	$\underline{-e_{234}}$	$\underline{-e_{125}}$	$\underline{e_{145}}$	$\underline{e_{123}}$	e_{1245}	$\underline{e_{12}}$	$\underline{-e_{1234}}$	$\underline{-e_{14}}$	$\underline{-e_{42}}$	$\underline{e_{124}}$
e_{23}	e_{23}	e_{123}	$\underline{e_{3}}$	$\underline{-e_{2}}$	e_{234}	e_{235}	$\underline{e_{42}}$	$\underline{-e_{53}}$	e_{2345}	$\underline{e_{12}}$	e_{1253}	$\underline{-e_{34}}$	$\underline{-e_{31}}$	$\underline{e_{25}}$	$\underline{1}$	e_{1234}	$\underline{e_{5}}$	Ω	$\underline{e_{134}}$	$\underline{-e_{245}}$	$\underline{-e_{125}}$	$\underline{e_{4}}$	$\underline{e_{345}}$	$\underline{e_{1}}$	$\underline{-e_{124}}$	$\underline{e_{135}}$	$\underline{e_{14}}$	$\underline{e_{51}}$	$\underline{-e_{3145}}$	$\underline{e_{1245}}$	$\underline{e_{45}}$	$\underline{e_{145}}$
e_{14}	e_{14}	$\underline{e_{4}}$	$-e_{124}$	$-e_{134}$	$\underline{-e_{1}}$	e_{145}	$\underline{-e_{31}}$	$-e_{1245}$	$\underline{e_{51}}$	$\underline{e_{34}}$	$\underline{-e_{45}}$	$\underline{-e_{12}}$	$\underline{e_{42}}$	$-e_{3145}$	e_{1234}	$\underline{1}$	Ω	$\underline{e_{5}}$	$\underline{-e_{2}}$	$\underline{e_{135}}$	$\underline{-e_{345}}$	$\underline{-e_{123}}$	$\underline{e_{125}}$	$\underline{e_{234}}$	$\underline{-e_{3}}$	$\underline{-e_{245}}$	$\underline{e_{23}}$	$\underline{-e_{2345}}$	$\underline{-e_{25}}$	$\underline{-e_{53}}$	$\underline{e_{1253}}$	$\underline{e_{235}}$
e_{235}	e_{235}	e_{1253}	$-e_{53}$	$-e_{25}$	$-e_{2345}$	e_{23}	$\underline{-e_{245}}$	$\underline{-e_{3}}$	$\underline{e_{234}}$	$\underline{e_{125}}$	$\underline{-e_{123}}$	$\underline{-e_{345}}$	$\underline{e_{135}}$	$\underline{-e_{2}}$	$\underline{e_{5}}$	Ω	$\underline{1}$	e_{1234}	$-e_{3145}$	$\underline{e_{42}}$	$\underline{-e_{12}}$	$\underline{-e_{45}}$	$\underline{e_{34}}$	$\underline{e_{51}}$	$-e_{1245}$	$\underline{-e_{31}}$	$\underline{e_{145}}$	$-e_{1}$	$\underline{-e_{134}}$	$\underline{-e_{124}}$	e_{4}	$\underline{e_{14}}$
e_{145}	e_{145}	e_{45}	e_{1245}	$-e_{3145}$	e_{51}	e_{14}	$\underline{e_{135}}$	$\underline{-e_{124}}$	$\underline{e_{1}}$	$\underline{e_{345}}$	$\underline{e_{4}}$	$\underline{-e_{125}}$	$\underline{-e_{245}}$	$\underline{e_{134}}$	Ω	$\underline{e_{5}}$	e_{1234}	$\underline{1}$	$\underline{e_{25}}$	$\underline{-e_{31}}$	$\underline{-e_{34}}$	e_{1253}	$\underline{e_{12}}$	e_{2345}	$\underline{-e_{53}}$	$\underline{e_{42}}$	$\underline{e_{235}}$	$\underline{e_{234}}$	$-e_{2}$	e_{3}	$\underline{e_{123}}$	$\underline{e_{23}}$
e_{124}	e_{124}	$-e_{42}$	$-e_{14}$	$-e_{1234}$	e_{12}	e_{1245}	$\underline{e_{123}}$	$\underline{e_{145}}$	$\underline{-e_{125}}$	$\underline{-e_{234}}$	$\underline{e_{245}}$	$\underline{-e_{1}}$	$\underline{e_{4}}$	Ω	$\underline{-e_{134}}$	$\underline{-e_{2}}$	$-e_{3145}$	$\underline{-e_{25}}$	$\underline{1}$	e_{1253}	$-e_{2345}$	$\underline{e_{31}}$	$\underline{-e_{51}}$	$\underline{-e_{34}}$	$\underline{e_{23}}$	$\underline{e_{45}}$	e_{3}	$\underline{e_{345}}$	$-e_{5}$	$\underline{-e_{235}}$	$\underline{-e_{135}}$	$\underline{e_{53}}$
e_{345}	e_{345}	e_{3145}	$-e_{2345}$	e_{45}	e_{53}	e_{34}	$\underline{e_{5}}$	$\underline{e_{234}}$	$\underline{e_{3}}$	$\underline{-e_{145}}$	$\underline{-e_{134}}$	$\underline{e_{235}}$	Ω	$\underline{e_{4}}$	$\underline{e_{245}}$	$\underline{-e_{135}}$	$\underline{-e_{42}}$	$\underline{e_{31}}$	e_{1253}	$\underline{1}$	$\underline{e_{14}}$	$\underline{-e_{25}}$	$\underline{-e_{23}}$	e_{1245}	$\underline{e_{51}}$	e_{1234}	$\underline{e_{125}}$	$\underline{e_{124}}$	$\underline{e_{123}}$	$-e_{1}$	e_{2}	$\underline{e_{12}}$
e_{135}	e_{135}	$-e_{53}$	$-e_{1253}$	e_{51}	e_{3145}	$-e_{31}$	$\underline{-e_{145}}$	$\underline{-e_{123}}$	$\underline{e_{134}}$	$\underline{-e_{5}}$	$\underline{e_{3}}$	Ω	$\underline{-e_{235}}$	$\underline{-e_{1}}$	$\underline{e_{125}}$	$\underline{e_{345}}$	$\underline{e_{12}}$	$\underline{e_{34}}$	$-e_{2345}$	$\underline{-e_{14}}$	$\underline{1}$	e_{1245}	$-e_{1234}$	$\underline{e_{25}}$	$\underline{-e_{45}}$	$\underline{-e_{23}}$	$\underline{-e_{245}}$	e_{2}	$\underline{e_{234}}$	$\underline{-e_{4}}$	$\underline{-e_{124}}$	$\underline{e_{42}}$
e_{234}	e_{234}	$-e_{1234}$	e_{34}	e_{42}	e_{23}	e_{2345}	$\underline{e_{2}}$	$\underline{-e_{345}}$	$\underline{-e_{235}}$	$\underline{e_{124}}$	Ω	$\underline{e_{3}}$	$\underline{e_{134}}$	$\underline{-e_{245}}$	$\underline{e_{4}}$	$\underline{e_{123}}$	$\underline{e_{45}}$	e_{1253}	$\underline{-e_{31}}$	$\underline{e_{25}}$	e_{1245}	$\underline{1}$	$\underline{e_{53}}$	$\underline{-e_{14}}$	$\underline{-e_{12}}$	e_{3145}	e_{1}	$\underline{e_{145}}$	$\underline{e_{135}}$	$\underline{e_{125}}$	$-e_{5}$	$\underline{e_{51}}$
e_{245}	e_{245}	$-e_{1245}$	e_{45}	e_{2345}	$-e_{25}$	$-e_{42}$	$\underline{e_{235}}$	$\underline{-e_{4}}$	$\underline{e_{2}}$	Ω	$\underline{-e_{124}}$	$\underline{-e_{5}}$	$\underline{e_{145}}$	$\underline{e_{234}}$	$\underline{-e_{345}}$	$\underline{-e_{125}}$	$\underline{-e_{34}}$	$\underline{-e_{12}}$	$\underline{e_{51}}$	$\underline{e_{23}}$	$-e_{1234}$	$\underline{-e_{53}}$	$\underline{1}$	e_{3145}	$-e_{1253}$	$\underline{e_{14}}$	$\underline{-e_{135}}$	$\underline{-e_{134}}$	e_{1}	$\underline{e_{123}}$	$-e_{3}$	$\underline{e_{31}}$
e_{123}	e_{123}	e_{23}	e_{31}	e_{12}	e_{1234}	$-e_{1253}$	$\underline{-e_{124}}$	$\underline{e_{135}}$	Ω	$\underline{e_{2}}$	$\underline{e_{235}}$	$\underline{-e_{134}}$	$\underline{e_{3}}$	$\underline{e_{125}}$	$\underline{e_{1}}$	$\underline{-e_{234}}$	$\underline{-e_{51}}$	e_{2345}	$\underline{e_{34}}$	e_{1245}	$\underline{-e_{25}}$	$\underline{e_{14}}$	e_{3145}	$\underline{1}$	$\underline{e_{42}}$	$\underline{-e_{53}}$	$-e_{4}$	e_{5}	$\underline{e_{345}}$	$\underline{e_{245}}$	$\underline{e_{145}}$	$\underline{e_{45}}$
e_{134}	e_{134}	e_{34}	e_{1234}	$-e_{14}$	$-e_{31}$	$-e_{3145}$	$\underline{e_{1}}$	Ω	$\underline{-e_{135}}$	$\underline{-e_{4}}$	$\underline{e_{345}}$	$\underline{e_{123}}$	$\underline{-e_{234}}$	$\underline{-e_{145}}$	$\underline{e_{124}}$	$\underline{-e_{3}}$	$-e_{1245}$	$\underline{e_{53}}$	$\underline{-e_{23}}$	$\underline{-e_{51}}$	$\underline{e_{45}}$	$\underline{e_{12}}$	$-e_{1253}$	$\underline{-e_{42}}$	$\underline{1}$	e_{2345}	$-e_{2}$	$\underline{-e_{245}}$	$\underline{-e_{235}}$	e_{5}	$\underline{e_{125}}$	$\underline{e_{25}}$
e_{125}	e_{125}	e_{25}	e_{51}	e_{1253}	$-e_{1245}$	e_{12}	Ω	$\underline{e_{1}}$	$\underline{e_{124}}$	$\underline{-e_{235}}$	$\underline{e_{2}}$	$\underline{e_{145}}$	$\underline{e_{5}}$	$\underline{-e_{123}}$	$\underline{-e_{135}}$	$\underline{e_{245}}$	$\underline{e_{31}}$	$\underline{-e_{42}}$	$\underline{-e_{45}}$	e_{1234}	$\underline{e_{23}}$	e_{3145}	$\underline{-e_{14}}$	$\underline{e_{53}}$	e_{2345}	$\underline{1}$	$\underline{e_{345}}$	$-e_{3}$	e_{4}	$\underline{e_{234}}$	$\underline{e_{134}}$	$\underline{e_{34}}$
e_{1234}	e_{1234}	$\underline{e_{234}}$	$\underline{-e_{134}}$	$\underline{e_{124}}$	$\underline{-e_{123}}$	Ω	$\underline{e_{12}}$	$\underline{e_{3145}}$	$\underline{e_{1253}}$	$\underline{-e_{42}}$	$\underline{-e_{2345}}$	$\underline{-e_{31}}$	$\underline{e_{34}}$	$\underline{-e_{1245}}$	$\underline{e_{14}}$	$\underline{e_{23}}$	$\underline{e_{145}}$	$\underline{e_{235}}$	$-e_{3}$	$\underline{e_{125}}$	$\underline{-e_{245}}$	$-e_{1}$	$\underline{-e_{135}}$	e_{4}	e_{2}	$\underline{e_{345}}$	1	$-e_{45}$	e_{53}	$-e_{25}$	e_{51}	e_{5}
e_{1253}	e_{1253}	$\underline{-e_{235}}$	$\underline{e_{135}}$	$\underline{-e_{125}}$	Ω	$\underline{e_{123}}$	$\underline{e_{1245}}$	$\underline{-e_{31}}$	$\underline{-e_{1234}}$	$\underline{-e_{25}}$	$\underline{e_{23}}$	$\underline{-e_{3145}}$	$\underline{e_{53}}$	$\underline{e_{12}}$	$\underline{e_{51}}$	$\underline{e_{2345}}$	e_{1}	$\underline{e_{234}}$	$\underline{e_{345}}$	$\underline{e_{124}}$	$-e_{2}$	$\underline{e_{145}}$	$\underline{-e_{134}}$	$-e_{5}$	$\underline{-e_{245}}$	e_{3}	e_{45}	1	$-e_{34}$	e_{42}	$-e_{14}$	e_{4}
e_{1245}	e_{1245}	$\underline{e_{245}}$	$\underline{-e_{145}}$	Ω	$\underline{e_{125}}$	$\underline{-e_{124}}$	$\underline{-e_{1253}}$	$\underline{-e_{14}}$	$\underline{e_{12}}$	$\underline{-e_{2345}}$	$\underline{e_{42}}$	$\underline{e_{51}}$	$\underline{e_{45}}$	$\underline{e_{1234}}$	$\underline{e_{3145}}$	$\underline{-e_{25}}$	$\underline{-e_{134}}$	e_{2}	e_{5}	$\underline{e_{123}}$	$\underline{e_{234}}$	$\underline{e_{135}}$	$-e_{1}$	$\underline{e_{345}}$	$\underline{-e_{235}}$	$-e_{4}$	$-e_{53}$	e_{34}	1	$-e_{23}$	e_{31}	e_{3}
e_{3145}	e_{3145}	$\underline{-e_{345}}$	Ω	$\underline{e_{145}}$	$\underline{-e_{135}}$	$\underline{e_{134}}$	$\underline{e_{51}}$	$\underline{-e_{1234}}$	$\underline{e_{31}}$	$\underline{e_{45}}$	$\underline{e_{34}}$	$\underline{e_{1253}}$	$\underline{e_{2345}}$	$\underline{-e_{14}}$	$\underline{-e_{1245}}$	$\underline{-e_{53}}$	$\underline{-e_{124}}$	$-e_{3}$	$\underline{-e_{235}}$	e_{1}	e_{4}	$\underline{e_{125}}$	$\underline{e_{123}}$	$\underline{e_{245}}$	$-e_{5}$	$\underline{e_{234}}$	e_{25}	$-e_{42}$	e_{23}	1	$-e_{12}$	e_{2}
e_{2345}	e_{2345}	Ω	$\underline{e_{345}}$	$\underline{-e_{245}}$	$\underline{e_{235}}$	$\underline{-e_{234}}$	$\underline{e_{25}}$	$\underline{e_{34}}$	$\underline{e_{23}}$	$\underline{e_{1245}}$	$\underline{e_{1234}}$	$\underline{-e_{53}}$	$\underline{-e_{3145}}$	$\underline{-e_{42}}$	$\underline{e_{45}}$	$\underline{-e_{1253}}$	$-e_{4}$	$\underline{e_{123}}$	$\underline{-e_{135}}$	$-e_{2}$	$\underline{-e_{124}}$	e_{5}	e_{3}	$\underline{e_{145}}$	$\underline{e_{125}}$	$\underline{e_{134}}$	$-e_{51}$	e_{14}	$-e_{31}$	e_{12}	1	e_{1}
Ω	Ω	e_{2345}	e_{3145}	e_{1245}	e_{1253}	e_{1234}	$\underline{e_{125}}$	$\underline{e_{134}}$	$\underline{e_{123}}$	$\underline{e_{245}}$	$\underline{e_{234}}$	$\underline{e_{135}}$	$\underline{e_{345}}$	$\underline{e_{124}}$	$\underline{e_{145}}$	$\underline{e_{235}}$	$\underline{e_{14}}$	$\underline{e_{23}}$	$\underline{e_{53}}$	$\underline{e_{12}}$	$\underline{e_{42}}$	$\underline{e_{51}}$	$\underline{e_{31}}$	$\underline{e_{45}}$	$\underline{e_{25}}$	$\underline{e_{34}}$	e_{5}	e_{4}	e_{3}	e_{2}	e_{1}	1

b (columns), a (rows)

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