The Orientation Congruent Algebra
Part I: A Nonassociative Clifford-Like Algebra [1]

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Abstract. The *correlated grade form* of twisted multivectors faithfully renders in symbols their native geometric structure. The discovery of this paper's nonassociative Clifford-like algebra was driven by trying to calculate exterior products of straight and twisted multivectors directly in a basis of blades (simple multivectors) in this form. The key was found to be the *orientation congruent* ($\mathcal{OC}$) algebra. This paper is Part I of a two part series. Part II (to be published later) will develop the motivating application just mentioned. In the first section we axiomatize the orientation congruent algebra by generators and relations. The next section derives the *sign factor function* σ and proves that the Clifford product times it is the multiplication of an explicitly Clifford-like algebra isomorphic to the orientation congruent algebra. Later sections show how to calculate the $\mathcal{OC}$ product in Mathematica and Clical; define the orientation congruent *contraction operators,* deduce their properties, derive other expressions for them, and use them to compute the $\mathcal{OC}$ product within the exterior algebra using a modified Cartan decomposition formula; develop the algebra's *product sequence graph* with labeled edges; explore the symmetries of and the matrices and orthogonal functions derivable from some forms of the Clifford and orientation congruent algebra multiplication tables; present a *predictor of a null associator* as a function of the grades of the three elements in it; state a conjecture on the *associomediative property* of *counits;* and develop matrix representations (under a *nonassociative* matrix product) of the orientation congruent product.

Dedication

to Elaine Yaw in honor of friendship

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Dedication

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1 Introduction

This paper is intended to be Part I of a two-part series on the orientation congruent ($\mathcal{OC}$) algebra and its motivating application. I am making both this first part (on the $\mathcal{OC}$ algebra per se) available for download from my website `http://felicity.freeshell.org` as well as the second part (on its application to computing the exterior products of straight and twisted multivectors).

This pair of papers has two aims:

Part I) to treat the $\mathcal{OC}$ algebra per se by developing the axioms, algebraic properties, multiplication table—in normal and graph-theoretic forms, multiplication algorithms, and matrix representations for the $\mathcal{OC}$ algebra;

Part II) to treat the $\mathcal{OC}$ algebra as applied by developing the *correlated grade form* and the generalized exterior product for straight and twisted multivectors represented in it.

This first paper begins in section 1 with the theoretical heart of the orientation congruent algebra, its axiomatic formulation. Unfortunately, this beginning reverses the natural course of development from motivating problem to general principles. A full exposition of the application that drove the discovery of the orientation congruent algebra will have to wait until Part II of this two-paper set. Nevertheless, we briefly discuss it in the next paragraph.

The orientation congruent (or $\mathcal{OC}$) algebra arose from the desire to *completely,* that is, pictorially, symbolically, and computationally, respect the native *inner and outer orientations* of the straight and twisted multicovectors (linear k-forms) making up electromagnetic fields. Some authors, first Schouten in Ref. [38], and then, following him, notably Burke in Refs. [12], [13], [14], and [15], and Jancewicz in Refs. [28] and [29] have shown how to respect the inner and outer orientations of such geometric objects in their graphical represen-taions. And Burke, with his so-called *William's twisted notation,* ([15], pp. 5f) came close to respecting the outer orientions of twisted differential forms in their symbolic representaion. The second paper will complete this trend by showing how to compute the exterior product of staight and twisted multivectors and multicovectors *directly* in a symbolic form faithfully representing their native inner and outer orientations. The author's discovery of the *correlated grade form* (*CGF*) to represent inner and outer oriented geometric objects and the $\mathcal{OC}$ algebra to compute their exterior product in this representation was essential to realizing this goal.

Some Conventions

In this paper, as is sufficient for an initial development, I limit the base vector space (upon which the larger vector spaces of the exterior, Clifford and orientation congruent algebras may be constructed) to a finite dimension over the reals $\mathbb{R}$. And, although I might easily adopt $\mathbb{R}^n$, the standard notation for a finite-dimensional vector space over $\mathbb{R}$, to symbolize the base vector space I prefer instead to use V^n. The symbol V^n has the disadvantage of requiring the awkward variation $V^n(\mathbb{R})$ to explicitly specify the scalar field. However, I make this choice because of Bossavit's warning[1] that the usual notation confuses a vector space V^n derived from physical modeling with a vector space $\mathbb{R}^n$ that is not only unnecessarily basis-dependent,[2] but also carries topological, metric, or other properties that may or may not apply to V^n.

The superscript in the symbol V^n indicates, of course, the dimension of the base space. Unless otherwise stated, the reader may assume the dimension of V appearing with no superscript to also be n.

Two more common Clifford algebra notations are reserved for other purposes in this series of papers. Therefore, the *reversion* of a multivector A, denoted with a tilde as $\widetilde{A}$ by P. Lounesto et al., is instead symbolized by a superscript dagger as $A^\dagger$ *à la* D. Hestenes et al. And the *grade involution* of A, often denoted with a hat as $\widehat{A}$ following Lounesto and other workers, is instead symbolized by the author's invention, a right hooked overline, as $\overline{A\rvert}$, or by the superscript symbol derived from it, the upper right "corner," as $A^{\urcorner}$. Other notations used in this paper that are standard or slightly modified are sometimes explained when introduced; those that are created for new concepts or that are designed to be more compact without becoming imprecise are, of course, always explained.

[1]See Ref. [7], p. 2, fn. 3, and some of his other works.

[2]The notation $\mathbb{R}^n$ is also commonly used for the vector space of all n-tuples which are the components of any vector in V^n with respect to some basis.

2 An Axiom System for the Orientation Congruent Algebra

In this section we first discuss the nondegenerate quadratic form $Q_{p,q}$, defining terms and notations for it and the two algebras of our interest associated with it. We then present a deductive foundation for the Clifford algebra $\mathcal{C}\ell_{p,q}$ of a nondegenerate quadratic form $Q_{p,q}$ in terms of *generators of and relations on* its elements(a *GR* axiom system for short). Next we give a similar axiomatic formulation for the orientation congruent algebra $\mathcal{OC}_{p,q}$ that is derived from the one for the corresponding Clifford algebra $\mathcal{C}\ell_{p,q}$ by modifying two of its axioms and adding one new one. Although we give only GR axiom sets in this first section of the paper, in the penultimate subsection we discuss some alternative axiomatic approaches. Finally, the last subsection presents the multiplication tables for some low order Clifford and orientation congruent algebras.

2.1 The Nondegenerate Quadratic Form $Q_{p,q}$ and Associated Algebras

Let us define the parallel relationships of the notations Q, $Q_{p,q}$, and Q_n; $\mathcal{C}\ell(Q)$, $\mathcal{C}\ell_{p,q}$, and $\mathcal{C}\ell_n$; and $\mathcal{OC}(Q)$, $\mathcal{OC}_{p,q}$, and $\mathcal{OC}_n$. Here n, p, and q are integers such that $n \geq 1$, and $p, q \geq 0$ with $p \geq 1$ or $q \geq 1$.First, we need the notions of a general *quadratic form* Q and its associated *symmetric bilinear form* B_Q.[3]

Definition 2.1 (Fauser [23], p. 3)
A quadratic form *on a vector space V^n over $\mathbb{R}$ is a map $Q\colon V^n \to \mathbb{R}$ such that*

$$Q(\alpha x) = \alpha^2 Q(x) \textit{ for all } \alpha \in \mathbb{R} \textit{ and } x \in V, \textit{ and} \tag{2.1a}$$

$$B_Q(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y)) \textit{ for all } x, y \in V^n, \tag{2.1b}$$

where $B_Q\colon V^n \times V^n \to \mathbb{R}$ is the symmetric bilinear form *associated with Q by the* polarization relation *given by eq. (2.1b).*

A quadratic form on V^n such that $Q(x) \neq 0$ for all $x \in V^n$ is said to be *nondegenerate*. Let Q be a nondegenerate quadratic form on V^n. If there exists an indexed set of mutually orthogonal[4] vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_p, \mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+q}\}$ for V^n such that for all $\mathbf{e}_i$

$$\begin{aligned} Q(\mathbf{e}_i) &> 0, \quad \text{for} \qquad 1 \leq i \leq p, \quad \text{and} \\ Q(\mathbf{e}_i) &< 0, \quad \text{for } p + 1 \leq i \leq p + q = n, \end{aligned}$$

[3] A map with two arguments such that $B\colon U \times V \to W$, where U, V, are W are vector spaces over $\mathbb{R}$, is said to be *bilinear* iff it is linear in both of its arguments. That is, $B(x + y, z) = B(x, z) + B(y, z)$, $B(x, y + z) = B(x, y) + B(x, z)$, and $B(\alpha x, \beta y) = \alpha\beta\, B(x, y)$ for all $\alpha, \beta \in \mathbb{R}$. The *form* part of its name means that for B_Q we have $W = \mathbb{R}$ in the definition of bilinearity just given. Also the word *symmetric* implies that $U = V$, since it means that $B_Q(x, y) = B_Q(y, x)$.

[4] That is, $B_Q(\mathbf{e}_i, \mathbf{e}_j) = 0$, if $i \neq j$. Note also that these vectors are not necessarily normalized to unit length.

we say that Q is of *signature* (p, q)[5] and we may write $Q_{p,q}$ to signify this. If $q = 0$, we then have $p = n$; whereupon we say Q is of *positive signature* n and we may write Q_n to indicate it. Also if $p = 0$, we then have $q = n$; whereupon we say Q is of *negative signature* n and we may write $Q_{0,n}$ to indicate that.

We may also represent that a nondegenerate quadratic form $Q_{p,q}$, Q_n, or $Q_{0,n}$ exists for the vector space V^n by writing $V^{p,q}$, $V^{n,0}$, or $V^{0,n}$, respectively.[6] We symbolize the corresponding Clifford algebras by $\mathcal{C}\ell_{p,q}$, $\mathcal{C}\ell_n$, and $\mathcal{C}\ell_{0,n}$; and the corresponding orientation congruent algebras by $\mathcal{OC}_{p,q}$, $\mathcal{OC}_n$, and $\mathcal{OC}_{0,n}$.[7] When discussing the Clifford or orientation congruent algebra of a general quadratic form Q, or when the signature (p, q) of Q is understood from context, we may also write $\mathcal{C}\ell(Q)$ or $\mathcal{OC}(Q)$, respectively. And when referring to the nondegenerate quadratic form of signature (p,q) associated with the Clifford algebra $\mathcal{C}\ell_{p,q}$, we will usually write simply Q instead of $Q_{p,q}$.

2.2 GR Axioms for the Clifford Algebra $\mathcal{C}\ell_{p,q}$ of a Nondegenerate Quadratic Form

Before we give a set of axioms for $\mathcal{OC}_{p,q}$ we first introduce a compact axiomatic definition of $\mathcal{C}\ell_{p,q}$ adapted from Lounesto's presentation.[8] Then we will expand this compact definition into a longer list of 15 axioms in three sets. Finally, after modifying this axiomatic formulation for $\mathcal{C}\ell_{p,q}$, we obtain a system of 16 axioms for $\mathcal{OC}_{p,q}$.

Hereafter the term *multivector* shall refer to any element of the Clifford algebra $\mathcal{C}\ell_{p,q}$ (or the orientation congruent algebra $\mathcal{OC}_{p,q}$) including those containing a scalar or vector component. Also the Clifford algebra product shall be denoted by an open dot $\circ$.[9]

Definition 2.2 (by Generators and Relations[10])
An associative algebra over $\mathbb{R}$ *with unit* 1 *is the* Clifford algebra $\mathcal{C}\ell_{p,q}$ *of a nondegenerate quadratic form* Q *on* V^n *(with the Clifford product symbolized by an open dot* $\circ$*) if it contains* V^n *and* $\mathbb{R} = \mathbb{R} \cdot 1$ *as distinct subspaces so that*

(1) $x \circ x = Q(x)$ *for all* $x \in V^n$*,*
(2) V^n *generates* $\mathcal{C}\ell_{p,q}$ *as an algebra over* $\mathbb{R}$*, and*
(3) $\mathcal{C}\ell_{p,q}$ *is not generated by any proper subset of* V^n*.*

[5] This (p, q) is, of course, the physicist's signature, not $s = p - q$, the mathematician's version.

[6] Note that we do not use V^n as a brief form of $V^{n,0}$, as we do for the corresponding notations for the Clifford and orientation congruent algebras, because we reserve V^n to indicate the vector space of dimension n that does not necessarily have a quadratic form associated with it.

[7] The degenerate *algebras* $\mathcal{C}\ell_{0,0}$ and $\mathcal{OC}_{0,0}$ also exist, but are not associated with a quadratic form since they are isomorphic with $\mathbb{R}$.

[8] See Ref. [33], pp. 190–2. Chapters 14, 21, and 22 of Ref. [33] also give several other definitions of a Clifford algebra.

[9] Usually Clifford multiplication is indicated by juxtaposition but here we prefer to distinguish between it and orientation congruent multiplication by giving each its own symbol: an open dot $\circ$, and a circled open dot $\circledcirc$, respectively.

As Lounesto remarks condition (3) of Def. 2.2 ensures that $\mathcal{C}\ell_{p,q}$ so defined is a *universal object* in the category theoretic sense and that the dimension of $\mathcal{C}\ell_{p,q}$ is 2^n. Roughly stated, the universality of an object means that it is unique up to isomorphism under a change of basis and that it is of the maximum size allowed by its definition.[11] Applied works commonly use a long set of axioms similar to those we give next to define the Clifford algebra $\mathcal{C}\ell_{p,q}$; however, usually their authors do not also mention the refinement of condition (3).

For reference and completeness we will now spell out Lounesto's axiomatic definition for the Clifford algebra $\mathcal{C}\ell_{p,q}$ of a nondegenerate quadratic form in a longer list of three sets of five axioms.[12] This list starts with two sets of five axioms which are the standard vector space axioms; however, now the vector space contains the multivectors in $\mathcal{C}\ell_{p,q}$ rather than just the vectors in the base space V^n. The first set of axioms given by Axiom Set I defines the properties of *multivector addition;* the second set given by Axiom Set II, the properties of two-sided *scalar multiplication.* Axiom Set III adds the last five axioms that define the algebraic properties of *Clifford multiplication.*

Axiom III.2 below assumes that $\mathbb{R} \subseteq \mathcal{C}\ell_{p,q}$; that is, that scalars are multivectors. Similarly, Axiom III.5 assumes that $V^n \subseteq \mathcal{C}\ell_{p,q}$; that is, that vectors are multivectors. However, in a more careful interpretation, one says that $\mathbb{R}$ and V^n are present in $\mathcal{C}\ell_{p,q}$ only as isomorphic images. The approach adopted here of identifying $\mathbb{R}$ and V^n with their images in $\mathcal{C}\ell_{p,q}$ creates redundancies in our axiom system that are discussed in detail in the footnotes. There we see that most of the axioms in the second set are subsumed and mirrored in those of the third set; scalar multiplication of multivectors will have become, after all, just Clifford multiplication by a scalar, and so must be consistent with it.

The first set of axioms defines the set of multivectors, $\mathcal{C}\ell_{p,q}$, as an *abelian group* under the operation of multivector addition. The *group operation* is written as an addition sign $+$.

[10] After Lounesto [33], p. 190.

[11] For a more detailed discussion of universality under the name *unique factorization property,* and in the context of the tensor product of vector spaces, see Shaw [40], pp. 274–7. See also Perwass's thesis [35], p. 18.

[12] These axioms for $\mathcal{C}\ell_{p,q}$ were adapted from those of Perwass [36], pp. 22–24. Shaw ([39], pp. 6,9) was also consulted for the vector space properties postulated in Axiom Set I. But note that this axiom system must be supplemented with conditions (2) and (3), and the requirement that $\mathbb{R}$ and V^n are distinct subspaces, all from Def. 2.2 due to Lounesto.

Axiom Set I Addition of Multivectors (in the Vector Space $\mathcal{C}\ell_{p,q}$)
There exists a binary operation called multivector addition, *symbolized by the* addition sign $+$, *and which is said to produce the* sum *of two multivectors, such that for all* $A, B, C \in \mathcal{C}\ell_{p,q}$

I.1.	$A + B \in \mathcal{C}\ell_{p,q}$,	*Existence and closure of sum*
I.2.	$A + B = B + A$,	*Commutativity*
I.3.	$(A + B) + C = A + (B + C)$,	*Associativity*
I.4.	$A + 0 = A$, *and*	*Existence of an identity*
I.5.	$A + (-A) = 0$.	*Existence of an inverse*[13]

In the second axiom set and below the elements of $\mathbb{R}$ are called *scalars*.

Axiom Set II Two-Sided Scalar Multiplication of Multivectors (in the Vector Space $\mathcal{C}\ell_{p,q}$)
There exists a binary operation called scalar multiplication, *symbolized by* juxtaposition, *such that for all* $A, B \in \mathcal{C}\ell_{p,q}$, $a \in V^n$, *and* $\alpha, \beta \in \mathbb{R}$

II.1.	$\alpha A, A\alpha \in \mathcal{C}\ell_{p,q}$,	*Existence and closure of scalar product*
II.2.	$A\alpha = \alpha A$,	*Commutativity*[14]
II.3.	$(\alpha\beta)A = \alpha(\beta A)$,	*Associativity of left scalar multiplication*
II.4.	$1A = A$,	*Existence of a left identity*
II.5a.	$(\alpha + \beta)A = \alpha A + \beta A$, *and*	*Distributivity of lsm*[15] *over scalar addition*
II.5b.	$\alpha(A + B) = \alpha A + \alpha B$.	*Distributivity of lsm over mv. addition*

Definition 2.3
An algebra *over* $\mathbb{R}$ *is a vector space* W *together with a bilinear*[16] *binary operation,* m, *called the algebra's* product *or* multiplication, *such that* $m\colon W \times W \to W$ *as* $m\colon (x, y) \mapsto m(x, y)$. *Sometimes* $m(x, y)$ *is written as* $x \textcircled{m} y$, *where* $\textcircled{m}$ *is usually some more abstract symbol such as* $\circ$, *or, simply by juxtaposing the arguments, as* xy.

Adding the third set of axioms turns the vector space $\mathcal{C}\ell_{p,q}$ into an associative algebra and relates the quadratic form Q associated with V^n to the Clifford square of the vectors in $\mathcal{C}\ell_{p,q}$. This algebra inherits a unit from the vector space by Axioms II.4 and III.2.

[13] This axiom is derivable from others in the two sets of vector space axioms and the field properties of $\mathbb{R}$ if we *define* $-A$ to be the result of the scalar multiplication $(-1)A$.

[14] Since $\mathbb{R}$ is a field, and thus has a *commutative* multiplication, it is not necessary to assume the existence of right scalar multiplication $A\alpha$ in Axiom II.1. Axiom II.2 may then be taken as a definition of right scalar multiplication as $A\alpha \equiv \alpha A$. See Shaw [39], p. 9, Rem. (b).

[15] We use "lsm" as short for "left scalar multiplication."

[16] A binary operation is bilinear iff it is linear in both of its arguments. Bilinearity implies distributivity of the product over vector space addition. Nevertheless, we explicitly include the distributive property in the axioms. For a more general defintion of bilinearity see fn. 3.

Axioms III.4 and III.5 have been placed at the end of the list because these two of the fifteen will be substantially modified for the orientation congruent algebra $\mathcal{OC}_{p,q}$.

Axiom Set III Clifford Multiplication of Multivectors (in the Algebra $\mathcal{C}\ell_{p,q}$)
There exists an algebraic product called Clifford multiplication, *symbolized by an* open dot $\circ$, *such that for all* $A, B, C \in \mathcal{C}\ell_{p,q}$, $a \in V^n$, *and* $\alpha \in \mathbb{R}$

III.1.	$A \circ B \in \mathcal{C}\ell_{p,q}$,	*Existence and closure of product*
III.2.	$\alpha \circ A = \alpha A, \quad A \circ \alpha = A\alpha$,	*Equality with l. & r. scalar mult.*[17]
III.3a.	$A \circ (B + C) = A \circ B + A \circ C$,	*Left distributivity over mv. add.*
III.3b.	$(B + C) \circ A = B \circ A + C \circ A$,	*Rt. distributivity over mv. add.*[18]
III.4.	$(A \circ B) \circ C = A \circ (B \circ C)$, *and*	*Associativity*[19]
III.5.	$a^2 \equiv a \circ a = Q(a)$.	*Equality of a the square of a vector and its quadratic form*

Implicit in the expressions of Axiom Set III is the usual parentheses-sparing convention of performing Clifford multiplications before performing multivector additions. Specifically, in Axioms III.3a and III.3b this operator precedence rule is applied on the right sides of the equations.

2.3 GR Axioms for the Orientation Congruent Algebra $\mathcal{OC}_{p,q}$ of a Nondegenerate Quadratic Form

We consider now another list of 15 axioms parallel to the one above, but modified. Then, we add one new axiom to obtain a list of 16 axioms[20] that will provide the axiomatic foundation for the $\mathcal{OC}_{p,q}$ algebra.

The first ten axioms in Axiom Sets I and II and the first three axioms in Axiom Set III are changed, but only trivially with the replacement of the terms and symbols referring to Clifford algebra with those referring to orientation congruent algebra. Therefore, we do not list the first ten of these axioms in their modified forms; however, we do list the first three axioms of Axiom Set III so changed. Next, we briefly describe the material changes and additions to Axiom Set III before making them.

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[17] As mentioned above, we have assumed that $\mathbb{R} \subseteq \mathcal{C}\ell(Q)$; that is, that scalars are multivectors. Therefore, the properties of scalar multiplication given in Axiom Set II are partially subsumed under those of Clifford multiplication given in this axiom set. In particular, this axiom and the one above it make Axiom II.1 redundant and it may be dropped.

[18] These Axioms III.3a and III.3b of the distributivity of Clifford multiplication, with the help of Axiom III.2, imply the (now redundant) Axioms II.5a and II.5b of the distributivity of left scalar multiplication.

[19] This Axiom III.4 of the associativity of Clifford multiplication, with the help of Axiom III.2, implies the (now redundant) Axiom II.3 of the associativity of scalar multiplication.

[20] As with that for $\mathcal{C}\ell_{p,q}$ this axiom system for $\mathcal{OC}_{p,q}$ must also be supplemented with suitably modified conditions similar to (2) and (3) of Def. 2.2 and the requirement that $\mathbb{R}$ and V^n are distinct subspaces, again all adapted from Lounesto ([33], p. 190).

The nontrivial changes required to the axioms in Axiom Set III are to

1) restrict the product used in Axiom III.4 for associativity from the orientation congruent product to the outer product,

2) extend the domain of Axiom III.5 for the equality of the algebra product square of a vector and its quadratic form to nonscalar blades,

3) add a new Axiom III.6′, supplementing the restricted Axiom III.4, that, in the given algebra or its extension by one dimension, requires the existence of a *counit* $\boldsymbol{\omega}_{\mathscr{A}}$ *of a set of multivectors* $\mathscr{A}$ with two key properties: $\mathscr{A}$*-universal commutativity* and *generalized commutativity of the right* $\boldsymbol{\omega}_{\mathscr{A}}$*-complement.*

All numbers of the modified axioms for the orientation congruent algebra will be marked with primes to indicated their correspondence with the original axioms for the Clifford algebra. For consistency the number of the new sixteenth axiom will also be primed.

Axiom Set III′ Orientation Congruent Multiplication of Multivectors (in the Algebra $\mathcal{OC}_{p,q}$)
There exists an algebraic product called orientation congruent multiplication, *and symbolized by an* circled open dot $\circledcirc$, *such that for all* $A, B, C \in \mathcal{OC}_{p,q}$ *and* $\alpha \in \mathbb{R}$

III.1′.	$A \circledcirc B \in \mathcal{OC}_{p,q}$,	*Existence and closure of product*
III.2′.	$\alpha \circledcirc A = \alpha A, \quad A \circledcirc \alpha = A\alpha$,	*Equality with l. & r. scalar mult.*
III.3a′.	$A \circledcirc (B + C) = A \circledcirc B + A \circledcirc C$,	*Left distributivity over mv. add.*
III.3b′.	$(B + C) \circledcirc A = B \circledcirc A + C \circledcirc A$.	*Rt. distributivity over mv. add.*

Before presenting the next axiom we pause to make some definitions which will also be used later.

Definition 2.4

a) A multivector $A \in \mathcal{OC}_{p,q}$ *is called an* r-blade *if, for some integer* $2 \leq r \leq n$*, it can be written as an orientation congruent multiproduct, with any grouping into binary products, of* r *mutually anticommuting vectors. That is,* $A = \boldsymbol{a}_1 \circledcirc \cdots \circledcirc \boldsymbol{a}_i \circledcirc \cdots \circledcirc \boldsymbol{a}_r$ *where all* $\boldsymbol{a}_i \in V^n$ *and* $\boldsymbol{a}_i \circledcirc \boldsymbol{a}_j = -\boldsymbol{a}_j \circledcirc \boldsymbol{a}_i$ *for all* $i \neq j$*. Note that we have used the convention that an unparenthesized multiproduct represents some arbitrary parenthesization of the multiproduct into binary products.*

b) We also define the term 1-blade *to mean* vector, *and the term* 0-blade *to mean* scalar. *And we interpret the multiproduct notation* $A = \boldsymbol{a}_1 \circledcirc \cdots \circledcirc \boldsymbol{a}_i \circledcirc \cdots \circledcirc \boldsymbol{a}_r$ *to be the vector* $A = \boldsymbol{a}_1$*, when* $r = 1$*, and some scalar* $A = \alpha$ *when* $r = 0$*.*

c) For any integer $0 \leq r \leq n$ *all zero-valued* r*-blades are considered to be equivalent. Thus,* 0 *represents a blade of indeterminate grade.*

d) An r-vector *is defined as a linear combination of* r*-blades.*

Definition 2.5

a) The outer product *of A_r and B_s, written with a* wedge $\wedge$, *is defined for any r-vector and s-vector $A_r, B_s \in \mathcal{OC}_{p,q}$ as the $(r+s)$-grade part of their orientation congruent product*

$$A_r \wedge B_s \equiv \langle A_r \circledcirc B_s \rangle_{r+s}. \tag{2.2}$$

b) The outer product of general multivectors $A, B \in \mathcal{OC}_{p,q}$ is then defined by

$$A \wedge B \equiv \sum_{r,s} \langle A \rangle_r \wedge \langle B \rangle_s = \sum_r \langle A \rangle_r \wedge B = \sum_s A \wedge \langle B \rangle_s. \tag{2.3}$$

Now we may continue with the next axiom.

Axiom Set III′ Orientation Congruent Multiplication (continued)
Orientation congruent multiplication determines through Def. 2.5 above the existence of the outer product as another algebraic product on the set $\mathcal{OC}_{p,q}$ such that for all $A, B, C \in \mathcal{OC}_{p,q}$

III.4′.	$(A \wedge B) \wedge C = A \wedge (B \wedge C)$.	*Associativity of outer product*

Orientation congruent multiplication is such that for any $A \in \mathcal{OC}_{p,q}$, with A the nonscalar r-blade $\boldsymbol{a}_1 \circledcirc \cdots \circledcirc \boldsymbol{a}_i \circledcirc \cdots \circledcirc \boldsymbol{a}_r$,

III.5′.	$A^2 \equiv A \circledcirc A =$ $Q(\boldsymbol{a}_1) \cdots Q(\boldsymbol{a}_i) \cdots Q(\boldsymbol{a}_r)$.	*Equality of the square of an r-blade and the product of the quadratic forms of its vectors*

The next axiom introduces the *counit.* The notions and notations of the following Def. 2.6 provide a naturally more compact way to write some expressions of Axiom III.6′ involving a counit.

Definition 2.6
We use a superscript $\boldsymbol{\omega}_{\mathscr{A}}$ attached to a multivector $A \in \mathcal{OC}_{p,q}$ to mean

$$A^{\boldsymbol{\omega}_{\mathscr{A}}} \equiv A \circledcirc \boldsymbol{\omega}_{\mathscr{A}}. \tag{2.4}$$

And, similarly, for left-sided multiplication by $\boldsymbol{\omega}_{\mathscr{A}}$ we define

$$^{\boldsymbol{\omega}_{\mathscr{A}}}\!A \equiv \boldsymbol{\omega}_{\mathscr{A}} \circledcirc A. \tag{2.5}$$

We call these operations right (left) $\boldsymbol{\omega}_{\mathscr{A}}$-complementation, *or* counit complementation by $\boldsymbol{\omega}_{\mathscr{A}}$, *and we give them precedence over orientation congruent, Clifford, and outer product multiplications.*

Axiom Set III′ Orientation Congruent Multiplication (finished)
Orientation congruent multiplication is such that within $\mathcal{OC}_{p,q}$, or its arbitrary extension by one dimension (which always exists),[21] *there exixts for all nonempty sets of multivectors $\mathscr{A}$ a (nonunique) nonscalar, unit magnitude blade called a* counit[22] *of $\mathscr{A}$, symbolized in boldface by $\boldsymbol{\omega}_{\mathscr{A}}$, such that for all (not necessarily distinct) $A, B \in \mathscr{A}$*

III.6a′.	$A \circledcirc \boldsymbol{\omega}_{\mathscr{A}} = \boldsymbol{\omega}_{\mathscr{A}} \circledcirc A$.	*$\mathscr{A}$-universal commutativity*
III.6b′.	$A^{\boldsymbol{\omega}_{\mathscr{A}}} \circledcirc B = A \circledcirc B^{\boldsymbol{\omega}_{\mathscr{A}}} =$ $(A \circledcirc B)^{\boldsymbol{\omega}_{\mathscr{A}}}$.	*Generalized commutativity of right $\boldsymbol{\omega}_{\mathscr{A}}$-complementation*

Applying the last axiom we make the following definitions.

Definition 2.7

a) If $\mathscr{A} = \mathcal{OC}_{p,q}$ *we call any* $\boldsymbol{\omega}_{\mathscr{A}} \in \mathcal{OC}_{p,q}$ "a" counit of the algebra $\mathcal{OC}_{p,q}$ *and we usually write such an* $\boldsymbol{\omega}_{\mathscr{A}}$ *using a boldface uppercase omega as* $\mathbf{\Omega}$. *We may then say that* $\mathcal{OC}_{p,q}$ *is a* perfect *orientation congruent (*$\mathcal{POC}$*) algebra and write it as* $\mathcal{POC}_{p,q}$.

b) In fact, if $n = p + q$ *is even there are no counits of* $\mathcal{OC}_{p,q}$. *Then we have only an* imperfect *orientation congruent (*$\mathcal{IOC}$*) algebra whose pseudoscalar does not satisfy Axiom III.6′. We symbolize such an algebra as* $\mathcal{IOC}_{p,q}$.[23] *We keep the words "orientation congruent," and the symbols* $\mathcal{OC}$ *and* $\mathcal{OC}_{p,q}$ *general so that they may refer to either case. When we do not know if an orientation congruent algebra has a counit or if we know it does not have one we will use the usual symbol* $\boldsymbol{I}$ *for the pseudoscalar.*

c) But, if n *is odd there are exactly two counits of the algebra that differ only by sign. These are* $\pm\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_i \wedge \cdots \wedge \mathbf{e}_n$ *for* $\mathbf{e}_i \in \mathscr{B} = \{\, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \,\}$, *where* $\mathscr{B}$ *is an ordered, orthonormal set of basis vectors for* V^n. *Choosing one of these counits establishes an* orientation *for* $\mathcal{OC}_{p,q}$. *An* $\mathbf{\Omega}$ *so chosen will be called* "the" counit of the algebra $\mathcal{OC}_{p,q}$, *or, simply,* the couint. *We may write it with an underline as* $\underline{\mathbf{\Omega}}$. *If an ordered, orthonormal set of basis vectors for* V^n *has been specified and no choice for the couint has been explicitly made, we assume that the counit is the one which is the product of the basis vectors in basis order* $\underline{\mathbf{\Omega}} = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n$.[24]

Precisely now with the presentation of this final axiom in Axiom Set III′ we have completed the construction a GR axiom system for the orientation congruent algebra of a nondegenerate quadratic form.

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[21] The author thanks John Browne [11] for suggesting that the even-dimensional spaces be included in the definition of an orientation congruent algebra.

[22] The name "counit" is a contraction of the phrase "coscalar unit." (However, when working with Hopf algebras or other areas of mathematics where the term *counit* is also used for an unrelated concept, one may employ the full phrase *coscalar unit.*) The "unit" part of the name is appropriate because a counit behaves algebraically like the unit. Indeed, for the set $\mathscr{A} = \mathcal{OC}_{p,q}$ 1 and -1 are the only elements other than $\mathbf{\Omega}$ and $-\mathbf{\Omega}$ (see the next Def. 2.7) that are of unit magnitude and have properties (a) and (b) of Axiom III.6′. And the "co" part of the name is consistent with the definition of a *coscalar* as an element of $\mathcal{OC}_{p,q}$ that has a *complementary grade* or *cograde* of $0 = n - k$ because it also has a grade of $k = n$ in the set of multivectors $\mathcal{OC}_{p,q}$ with $n = p + q$. Generally, when working in the algebra $\mathcal{OC}_{p,q}$, a minimal grade counit $\boldsymbol{\omega}_{\mathscr{A}}$ of a nonempty set of multivectors $\mathscr{A}$ has a cograde of $0 = m - k$ (or a grade of $k = m$) *relative to* the smallest odd $m = r + s$ such that $\mathscr{A} \subseteq \mathcal{OC}_{r,s} \subseteq \mathcal{OC}_{p,q}$.

[23] As we have postulated an $\mathcal{IOC}_{r,s}$ with $m = r + s$ can always be extended to an $\mathcal{POC}_{p,q}$ with $n = m + 1 = p + q$ and having primed basis vectors by adding another basis vector $\mathbf{e}_{m+1}{}' = \mathbf{e}_n{}'$ (making $p = r$ and $q = s + 1$) or $\mathbf{e}_{r+1}{}' = \mathbf{e}_p{}'$ (making $p = r + 1$ and $q = s$) in a signature-ordered, orthogonal set of basis vectors. An $\mathcal{IOC}$ algebra is actually a *subalgebra* of the next higher-dimensional $\mathcal{POC}$ algebra.

[24] In this case the counit $\underline{\mathbf{\Omega}}$ is the same element in $\mathcal{OC}_{p,q}$ as what is called, in the language of *geometric algebra* (Clifford algebra given a geometric interpretation), the *unit pseudoscalar* I associated with an orthonormal *frame* (set of basis vectors) for $\mathcal{C}\ell_{p,q}$. Also, the q part of the signature (p, q) of the quadratic form of $\mathcal{OC}_{p,q}$ determines the sign of the orientation congruent square of a counit of the algebra by $\mathbf{\Omega}^2 = (-\mathbf{\Omega})^2 = (-1)^q$.

Remarks 2.1

a) The last Axiom III.6′ along with Axiom III.4′ replaces Axiom III.4 expressing the associativity of the Clifford product. Associativity is just one member of the class of possible "bracket shifting rules."

b) Axiom III.4′ partially replaces the general associativity of the Clifford product with that of the outer product. The outer product is derived by a grade selection from the orientation congruent product. Equivalently, this axiom may be restated to postulate the associativity of the $\mathcal{OC}$ product of the component vectors of two blades if those component vectors mutually anticommute under the orientation congruent product when combined as one group. This axiom has a direct analog as a theorem in all Clifford algebras $\mathcal{C}\ell_{p,q}$.

c) Axiom III.6′ supplements Axiom III.4′ with a pair of commutative and bracket shifting rules both involving $\boldsymbol{\omega}_{\mathscr{A}}$, and both more complicated, but generally applicable. Axiom III.4′ has a direct analog as a theorem in all Clifford algebras. But part (a) of Axiom III.6′ of has a direct analog as a theorem in only Clifford algebras $\mathcal{C}\ell_{p,q}$ with odd $n = p + q$.

d) In summary, we might say that to transform the axioms for $\mathcal{C}\ell_{p,q}$ into those for $\mathcal{OC}_{p,q}$ we have traded an expansion of the domain of applicability of Axiom III.5 from vectors to blades in Axiom III.5′ for a restriction of the domain of applicability of Axiom III.4 with its consequent fragmentation into the two Axioms III.4′, III.6′.

2.4 Other Axiom Systems

The literature provides other axiomatic formulations of Clifford algebras of varying generality. Here we will consider their adaptability to the orientation congruent algebra.[25]

These other Clifford algebra axiom systems range, for example, from those describing a Clifford algebra as an ideal of a tensor algebra ([33], pp. 193f), or describing it in category-theoretic terms as the universal object of a quadratic algebra ([33], pp. 192f), or embedding it as a subalgebra of the associated exterior algebra's endomorphism algebra through the Chevalley-operator representation (which Chevalley [18] based on the Cartan decomposition formula),[26] or describing it as a Hopf gebra[27] using tensor algebra and category theory expressed in commutative and tangle diagrams ([24], chs. 3– 5), to providing a multiplication rule for basis blades represented by n-tuples of binary digits called multi-indices ([33], ch. 21).[28]

[25] Subsection 5.1 has more remarks on axiomatizations.

[26] This decomposition formula is credited to E. Cartan by Crumeyrolle ([20], p. 44) and Abłamowicz ([2] p. 463). Chevalley's method is also used by Lounesto ([33], ch. 22), Crumeyrolle ([20], p. 45), and Oziewicz [34]. It is also implicit in the paper of Fernández, Moya, and Rodrigues ([25], p. 15). Also see subsection 5.1 for more remarks on axiomatizations.

[27] This is not a misprint. Without going into details, a Hopf gebra is a more general structure than a Hopf algebra ([24], p. 65).

[28] This last is really a specialized form of GR axiomitization.

Only three of these approaches to the axiomatization of Clifford algebra are directly convertible to the orientation congruent algebra. One is the definition as a universal object of quadratic algebras. The modification required is simply using nonassociative quadratic algebras in place of the (assumed) associative quadratic algebras and adding other relations to represent Axioms III.4′, III.5′, and III.6′. However, since this very abstract definition is nonconstructive, it is not useful for calculating the orientation congruent product.

It is only the last two definitions, one based on Hopf gebra and the other on a multiplication rule for basis blades that are both adaptable and useful. That is because the other approaches are based on intrinsically associative algebras. Hopf gebras, however, are not ruled out; associativity is not necessary for their definition ([24], p. 65). Also as demonstrated by Fauser [24] the Hopf gebraic approach is very fruitful in producing grade-free computational algorithms for very general forms of Clifford algebras.

The last definition from a multiplication rule for basis blades is easily generalizable to Clifford-like algebras. These are essentially the algebras of the Clifford product but as modified by a sign rule that may differ from the standard Clifford algebra one ([33], pp. 284ff). The Clifford-like algebras, however, are not necessarily associative. They may also have other properties that vary from those of the Clifford algebras. In the following section we will construct the explicitly Clifford-like *sigma orientation congruent algebra* $\sigma\mathcal{OC}_{p,q}$. As suggested above we will fashion the product of the sigma orientation congruent algebra from the Clifford product times a *sign factor function* σ.

In section 3 we also prove the deductive equivalence of the set of primed axioms for the orientation congruent algebra $\mathcal{OC}_{p,q}$ with that of the unprimed axioms for the Clifford algebra $\mathcal{C}\ell_{p,q}$ supplemented by an existence axiom for the sigma orientation congruent product. In so doing we establish that the sigma orientation congruent algebra of a nondegenerate quadratic form is isomorphic to the corresponding orientation congruent algebra. Then, instead of reasoning directly from the axioms of the current section, we can also prove theorems for the orientation congruent algebra by interpreting its product as the sigma orientation congruent product and manipulating ordinary algebraic expressions derived from the sign factor function while citing verified Clifford algebra theorems.

Actually, in the sequel to section 3 the sigma form of the orientation congruent product will be the basis for investigating the $\mathcal{OC}_{p,q}$ algebra. Indeed, in section 3 while simply proving the equivalence of the orientation congruent product and the sigma orientation congruent product other proofs of some assertions made in this section will naturally fall out as byproducts. One statement with such an incidental proof is that a perfect orientation congruent algebra $\mathcal{POC}_{p,q}$ exists in all and only those base spaces V^n of odd dimension, or, complementarily, that an imperfect orientation congruent algebra $\mathcal{IOC}_{p,q}$ exists in all and only those base spaces V^n of even dimension.

2.5 Multiplication Tables

We end with the multiplication tables for the Clifford algebra $\mathcal{C}\ell_3$ (Tab. 2.1), and the orientation congruent algebras $\mathcal{OC}_3$ (Tab. 2.2) and $\mathcal{OC}_5$ (Tab. 2.5). In these tables the basis blades are written with multi-indices so that, for example, $\mathbf{e}_{23} = \mathbf{e}_2 \circ \mathbf{e}_3$ or $\mathbf{e}_2 \circledcirc \mathbf{e}_3$ depending on which algebra appears in the table. Also the counits of the two orientation congruent algebras are written in omega notation as $\boldsymbol{\Omega} = \mathbf{e}_{123} = \mathbf{e}_1 \circledcirc \mathbf{e}_2 \circledcirc \mathbf{e}_3$ or $\boldsymbol{\Omega} = \mathbf{e}_{12345} = \mathbf{e}_1 \circledcirc \mathbf{e}_2 \circledcirc \mathbf{e}_3 \circledcirc \mathbf{e}_4 \circledcirc \mathbf{e}_5$ depending on the algebra.[29] We write the pseudoscalar of $\mathcal{C}\ell_3$ as $\boldsymbol{I} = \mathbf{e}_{123} = \mathbf{e}_1 \circ \mathbf{e}_2 \circ \mathbf{e}_3$.

The underlined entries in the orientation congruent algebra multiplication tables are oppositely signed compared to those in the tables for the corresponding Clifford algebras. Also in all tables the entries in red-colored cells are negatively signed; while the entries in white-colored cells are positively signed.

Tabs. 2.2 and 2.5 show a certain form of the multiplication tables for the algebras $\mathcal{OC}_3$ and $\mathcal{OC}_5$. The cell coloring in these tables makes the reflection symmetry of the signs of the products about the central horizontal and vertical axes easy to see. Tab. 2.1 shows the same form of the multiplication table for the Clifford algebra $\mathcal{C}\ell_3$. Here the pattern of cell coloring has no obvious symmetry.

Both the reflection symmetries in Tabs. 2.2 and 2.5 and their lack in Tab. 2.1 result from displaying these tables in a canonical form specific to the orientation congruent algebra. The arrangement of these tables is an example of a *multiplication table canonical form* (MTCF) of type *OC1*.

Any MTCF for an algebra is determined by just two criteria: 1) the ordering chosen for the multi-indices of each basis blade; and 2) the ordering of the basis blades in the indicial leftmost column and top row of the table. Because the full definition of a MTCF of type OC1 is rather complicated we defer it to a later section. However, we may roughly say that a type OC1 MTCF satisfies criterion 1) by ordering the multi-indices of the basis blades so that as a set they are *coherently oriented* (in a specific way) relative to the couinit $\underline{\boldsymbol{\Omega}}$. Also we may roughly say that it satifies criterion 2) by placing the factor basis blades in the indicial column and row in a kind of *graded, reflected complementary order.* As the “1” in “OC1” suggests these two requirements define just one of several related multiplication table canonical forms.

For Clifford algebras we can define a MTCF of type *CL1* that is *increasing numerical order* within the multi-index sequences of each basis blade and *Gray code order* ([33], pp. 281ff) for the factor basis blades in the indicial column and row. If a Clifford algebra multiplication table is in CL1 canonical form, the signs of the products display reflection symmetry about the central vertical axis just as they do for an orientation congruent algebra multiplication table in OC1 form. However, the second sign symmetry pattern differs: it becomes vertical *translation* symmetry between adjacent rows paired off starting from the first. And now it is the $\mathcal{OC}_3$ multiplication table in CL1 canonical form whose product signs display no obvious symmetries. Further discussion is deferred until §7.

[29] Because it conflicts with another usage in these tables (defined in the next paragraph) we have forgone the underlining of these omegas to symbolize *the* counits of these $\mathcal{OC}$ algebras (the convention established in Def. 2.7 of the previous section).

					b				
	$a \circ b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$
	1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$
	$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$\boldsymbol{I}$	$\mathbf{e}_{23}$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_1$	$\boldsymbol{I}$	$\mathbf{e}_3$	$\mathbf{e}_{31}$
	$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\boldsymbol{I}$	$\mathbf{e}_1$	$-\mathbf{e}_2$	$\mathbf{e}_{12}$
a	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\boldsymbol{I}$	-1	$\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_3$
	$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$\boldsymbol{I}$	$-\mathbf{e}_1$	$-\mathbf{e}_{23}$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_2$
	$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	-1	$-\mathbf{e}_1$
	$\boldsymbol{I}$	$\boldsymbol{I}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	-1

Table 2.1: The Multiplication Table for the Clifford Algebra $\mathcal{C}\ell_3$. The factors are in graded, reflected complementary order. Their indices are ordered so that the the basis blades have coherent orientations. Red cells contain negative entries.

					b				
	$a \circledcirc b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
	1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
	$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\underline{-\mathbf{e}_2}$	$\underline{\mathbf{e}_3}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$\underline{\mathbf{e}_1}$	$\boldsymbol{\Omega}$	$\underline{-\mathbf{e}_3}$	$\mathbf{e}_{31}$
	$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\boldsymbol{\Omega}$	$\underline{-\mathbf{e}_1}$	$\underline{\mathbf{e}_2}$	$\mathbf{e}_{12}$
a	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\underline{\mathbf{e}_2}$	$\underline{-\mathbf{e}_1}$	$\boldsymbol{\Omega}$	$\underline{1}$	$\underline{-\mathbf{e}_{23}}$	$\underline{\mathbf{e}_{31}}$	$\underline{\mathbf{e}_3}$
	$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\underline{-\mathbf{e}_3}$	$\boldsymbol{\Omega}$	$\underline{\mathbf{e}_1}$	$\underline{\mathbf{e}_{23}}$	$\underline{1}$	$\underline{-\mathbf{e}_{12}}$	$\underline{\mathbf{e}_2}$
	$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$	$\underline{\mathbf{e}_3}$	$\underline{-\mathbf{e}_2}$	$\underline{-\mathbf{e}_{31}}$	$\underline{\mathbf{e}_{12}}$	$\underline{1}$	$\underline{\mathbf{e}_1}$
	$\boldsymbol{\Omega}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\underline{\mathbf{e}_3}$	$\underline{\mathbf{e}_2}$	$\underline{\mathbf{e}_1}$	$\underline{1}$

Table 2.2: The Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_3$. The same orderings of factors and their indices are used in this table as for the corresponding Clifford algebra $\mathcal{C}\ell_3$ in Tab. 2.1. Red cells contain negative entries. The underlined entries are oppositely signed compared to those in Tab. 2.1.

$a \circ b$	1	$\mathbf{e}_1$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$\mathbf{e}_{23}$	$\boldsymbol{I}$	$\mathbf{e}_{13}$	$\mathbf{e}_3$
1	1	$\mathbf{e}_1$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$\mathbf{e}_{23}$	$\boldsymbol{I}$	$\mathbf{e}_{13}$	$\mathbf{e}_3$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_2$	$\mathbf{e}_{12}$	$\boldsymbol{I}$	$\mathbf{e}_{23}$	$\mathbf{e}_3$	$\mathbf{e}_{13}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	-1	$\mathbf{e}_1$	$\mathbf{e}_{13}$	$-\mathbf{e}_3$	$-\mathbf{e}_{23}$	$\boldsymbol{I}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	$-\mathbf{e}_1$	1	$\mathbf{e}_3$	$-\mathbf{e}_{13}$	$-\boldsymbol{I}$	$\mathbf{e}_{23}$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$	$-\mathbf{e}_{13}$	$-\mathbf{e}_3$	-1	$-\mathbf{e}_1$	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\boldsymbol{I}$	$\boldsymbol{I}$	$\mathbf{e}_{23}$	$-\mathbf{e}_3$	$-\mathbf{e}_{13}$	$-\mathbf{e}_1$	-1	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_{13}$	$\mathbf{e}_{13}$	$-\mathbf{e}_3$	$\mathbf{e}_{23}$	$-\boldsymbol{I}$	$-\mathbf{e}_{12}$	$\mathbf{e}_2$	-1	$\mathbf{e}_1$
$\mathbf{e}_3$	$\mathbf{e}_3$	$-\mathbf{e}_{13}$	$\boldsymbol{I}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_2$	$\mathbf{e}_{12}$	$-\mathbf{e}_1$	1

Table 2.3: The Multiplication Table for the Clifford Algebra $\mathcal{C}\ell_3$. The factors are in Gray code order. Their indices are in increasing numerical order. Red cells contain negative entries.

$a \circledcirc b$	1	$\mathbf{e}_1$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{13}$	$\mathbf{e}_3$
1	1	$\mathbf{e}_1$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{13}$	$\mathbf{e}_3$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\underline{-\mathbf{e}_2}$	$\mathbf{e}_{12}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$	$\underline{-\mathbf{e}_3}$	$\mathbf{e}_{13}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\underline{\mathbf{e}_2}$	$\underline{1}$	$\underline{-\mathbf{e}_1}$	$\underline{-\mathbf{e}_{13}}$	$\underline{\mathbf{e}_3}$	$\underline{\mathbf{e}_{23}}$	$\boldsymbol{\Omega}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	$\underline{\mathbf{e}_1}$	1	$\underline{-\mathbf{e}_3}$	$-\mathbf{e}_{13}$	$-\boldsymbol{\Omega}$	$\mathbf{e}_{23}$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$	$\underline{\mathbf{e}_{13}}$	$\underline{\mathbf{e}_3}$	$\underline{1}$	$\underline{\mathbf{e}_1}$	$\underline{-\mathbf{e}_{12}}$	$\underline{-\mathbf{e}_2}$
$\boldsymbol{\Omega}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$	$\underline{\mathbf{e}_3}$	$-\mathbf{e}_{13}$	$\underline{\mathbf{e}_1}$	$\underline{1}$	$\underline{-\mathbf{e}_2}$	$\mathbf{e}_{12}$
$\mathbf{e}_{13}$	$\mathbf{e}_{13}$	$\underline{\mathbf{e}_3}$	$\underline{-\mathbf{e}_{23}}$	$-\boldsymbol{\Omega}$	$\underline{\mathbf{e}_{12}}$	$\underline{-\mathbf{e}_2}$	$\underline{1}$	$\underline{-\mathbf{e}_1}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$-\mathbf{e}_{13}$	$\boldsymbol{\Omega}$	$-\mathbf{e}_{23}$	$\underline{\mathbf{e}_2}$	$\mathbf{e}_{12}$	$\underline{\mathbf{e}_1}$	1

Table 2.4: The Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_3$. The same orderings of factors and their indices are used in this table as for the corresponding Clifford algebra $\mathcal{C}\ell_3$ in Tab. 2.3. Red cells contain negative entries. The underlined entries are oppositely signed compared to those in Tab. 2.3.

Table 2.5: The Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_5$

b

a

$a \circledcirc b$	1	e_{1}	e_{2}	e_{3}	e_{4}	e_{5}	e_{34}	e_{25}	e_{45}	e_{31}	e_{51}	e_{42}	e_{12}	e_{53}	e_{23}	e_{14}	e_{235}	e_{145}	e_{124}	e_{345}	e_{135}	e_{234}	e_{245}	e_{123}	e_{134}	e_{125}	e_{1234}	e_{1253}	e_{1245}	e_{3145}	e_{2345}	Ω
1	1	e_{1}	e_{2}	e_{3}	e_{4}	e_{5}	e_{34}	e_{25}	e_{45}	e_{31}	e_{51}	e_{42}	e_{12}	e_{53}	e_{23}	e_{14}	e_{235}	e_{145}	e_{124}	e_{345}	e_{135}	e_{234}	e_{245}	e_{123}	e_{134}	e_{125}	e_{1234}	e_{1253}	e_{1245}	e_{3145}	e_{2345}	Ω
e_{1}	e_{1}	1	e_{12}	$-e_{31}$	e_{14}	$-e_{51}$	e_{134}	e_{125}	e_{145}	$\underline{e_{3}}$	$\underline{e_{5}}$	$-e_{124}$	$\underline{-e_{2}}$	$-e_{135}$	e_{123}	$\underline{-e_{4}}$	$-e_{1253}$	e_{45}	$-e_{42}$	$-e_{3145}$	$-e_{53}$	e_{1234}	e_{1245}	e_{23}	e_{34}	e_{25}	$\underline{-e_{234}}$	$\underline{e_{235}}$	$\underline{-e_{245}}$	$\underline{e_{345}}$	Ω	e_{2345}
e_{2}	e_{2}	$-e_{12}$	1	e_{23}	$-e_{42}$	e_{25}	e_{234}	$\underline{-e_{5}}$	e_{245}	e_{123}	e_{125}	$\underline{e_{4}}$	$\underline{e_{1}}$	$-e_{235}$	$\underline{-e_{3}}$	$-e_{124}$	$-e_{53}$	$-e_{1245}$	$-e_{14}$	e_{2345}	e_{1253}	e_{34}	e_{45}	e_{31}	$-e_{1234}$	e_{51}	$\underline{e_{134}}$	$\underline{-e_{135}}$	$\underline{e_{145}}$	Ω	$\underline{-e_{345}}$	e_{3145}
e_{3}	e_{3}	e_{31}	$-e_{23}$	1	e_{34}	$-e_{53}$	$\underline{-e_{4}}$	$-e_{235}$	e_{345}	$\underline{-e_{1}}$	e_{135}	e_{234}	e_{123}	$\underline{e_{5}}$	$\underline{e_{2}}$	$-e_{134}$	$-e_{25}$	e_{3145}	e_{1234}	e_{45}	e_{51}	e_{42}	$-e_{2345}$	e_{12}	$-e_{14}$	$-e_{1253}$	$\underline{-e_{124}}$	$\underline{e_{125}}$	Ω	$\underline{-e_{145}}$	$\underline{e_{245}}$	e_{1245}
e_{4}	e_{4}	$-e_{14}$	e_{42}	$-e_{34}$	1	e_{45}	$\underline{e_{3}}$	$-e_{245}$	$\underline{-e_{5}}$	$-e_{134}$	e_{145}	$\underline{-e_{2}}$	e_{124}	e_{345}	e_{234}	$\underline{e_{1}}$	e_{2345}	e_{51}	e_{12}	e_{53}	$-e_{3145}$	e_{23}	$-e_{25}$	$-e_{1234}$	$-e_{31}$	e_{1245}	$\underline{e_{123}}$	Ω	$\underline{-e_{125}}$	$\underline{e_{135}}$	$\underline{-e_{235}}$	e_{1253}
e_{5}	e_{5}	e_{51}	$-e_{25}$	e_{53}	$-e_{45}$	1	e_{345}	$\underline{e_{2}}$	$\underline{e_{4}}$	$-e_{135}$	$\underline{-e_{1}}$	$-e_{245}$	e_{125}	$\underline{-e_{3}}$	e_{235}	e_{145}	e_{23}	e_{14}	$-e_{1245}$	e_{34}	$-e_{31}$	$-e_{2345}$	$-e_{42}$	e_{1253}	e_{3145}	e_{12}	Ω	$\underline{-e_{123}}$	$\underline{e_{124}}$	$\underline{-e_{134}}$	$\underline{e_{234}}$	e_{1234}
e_{34}	e_{34}	e_{134}	e_{234}	$\underline{e_{4}}$	$\underline{-e_{3}}$	e_{345}	$\underline{1}$	e_{2345}	$\underline{e_{53}}$	$\underline{-e_{14}}$	e_{3145}	$\underline{e_{23}}$	e_{1234}	$\underline{-e_{45}}$	$\underline{-e_{42}}$	$\underline{e_{31}}$	$\underline{e_{245}}$	$\underline{-e_{135}}$	$\underline{-e_{123}}$	$\underline{e_{5}}$	$\underline{e_{145}}$	$\underline{e_{2}}$	$\underline{-e_{235}}$	$\underline{e_{124}}$	$\underline{e_{1}}$	Ω	$\underline{e_{12}}$	$\underline{-e_{1245}}$	$\underline{e_{1253}}$	$\underline{e_{51}}$	$\underline{e_{25}}$	$\underline{e_{125}}$
e_{25}	e_{25}	e_{125}	$\underline{e_{5}}$	$-e_{235}$	$-e_{245}$	$\underline{-e_{2}}$	e_{2345}	$\underline{1}$	$\underline{-e_{42}}$	$-e_{1253}$	$\underline{e_{12}}$	$\underline{e_{45}}$	$\underline{-e_{51}}$	$\underline{-e_{23}}$	$\underline{e_{53}}$	$-e_{1245}$	$\underline{-e_{3}}$	$\underline{e_{124}}$	$\underline{-e_{145}}$	$\underline{-e_{234}}$	$\underline{e_{123}}$	$\underline{e_{345}}$	$\underline{-e_{4}}$	$\underline{-e_{135}}$	Ω	$\underline{e_{1}}$	$\underline{-e_{3145}}$	$\underline{-e_{31}}$	$\underline{-e_{14}}$	$\underline{e_{1234}}$	$\underline{e_{34}}$	$\underline{e_{134}}$
e_{45}	e_{45}	e_{145}	e_{245}	e_{345}	$\underline{e_{5}}$	$\underline{-e_{4}}$	$\underline{-e_{53}}$	$\underline{e_{42}}$	$\underline{1}$	e_{3145}	$\underline{e_{14}}$	$\underline{-e_{25}}$	e_{1245}	$\underline{e_{34}}$	e_{2345}	$\underline{-e_{51}}$	$\underline{-e_{234}}$	$\underline{e_{1}}$	$\underline{e_{125}}$	$\underline{e_{3}}$	$\underline{-e_{134}}$	$\underline{e_{235}}$	$\underline{e_{2}}$	Ω	$\underline{e_{135}}$	$\underline{-e_{124}}$	$\underline{-e_{1253}}$	$\underline{e_{1234}}$	$\underline{e_{12}}$	$\underline{e_{31}}$	$\underline{e_{23}}$	$\underline{e_{123}}$
e_{31}	e_{31}	$\underline{-e_{3}}$	e_{123}	$\underline{e_{1}}$	$-e_{134}$	$-e_{135}$	$\underline{e_{14}}$	$-e_{1253}$	e_{3145}	$\underline{1}$	$\underline{-e_{53}}$	$-e_{1234}$	$\underline{e_{23}}$	$\underline{e_{51}}$	$\underline{-e_{12}}$	$\underline{-e_{34}}$	$\underline{-e_{125}}$	$\underline{-e_{345}}$	$\underline{e_{234}}$	$\underline{e_{145}}$	$\underline{-e_{5}}$	$\underline{-e_{124}}$	Ω	$\underline{e_{2}}$	$\underline{-e_{4}}$	$\underline{e_{235}}$	$\underline{-e_{42}}$	$\underline{-e_{25}}$	$\underline{e_{2345}}$	$\underline{e_{45}}$	$\underline{-e_{1245}}$	$\underline{e_{245}}$
e_{51}	e_{51}	$\underline{-e_{5}}$	e_{125}	e_{135}	e_{145}	$\underline{e_{1}}$	e_{3145}	$\underline{-e_{12}}$	$\underline{-e_{14}}$	$\underline{e_{53}}$	$\underline{1}$	e_{1245}	$\underline{e_{25}}$	$\underline{-e_{31}}$	e_{1253}	$\underline{e_{45}}$	$\underline{e_{123}}$	$\underline{e_{4}}$	$\underline{-e_{245}}$	$\underline{e_{134}}$	$\underline{e_{3}}$	Ω	$\underline{e_{124}}$	$\underline{-e_{235}}$	$\underline{-e_{345}}$	$\underline{e_{2}}$	$\underline{e_{2345}}$	$\underline{e_{23}}$	$\underline{e_{42}}$	$\underline{e_{34}}$	$\underline{-e_{1234}}$	$\underline{e_{234}}$
e_{42}	e_{42}	$-e_{124}$	$\underline{-e_{4}}$	e_{234}	$\underline{e_{2}}$	$-e_{245}$	$\underline{-e_{23}}$	$\underline{-e_{45}}$	$\underline{e_{25}}$	$-e_{1234}$	e_{1245}	$\underline{1}$	$\underline{-e_{14}}$	$-e_{2345}$	$\underline{e_{34}}$	$\underline{e_{12}}$	$\underline{e_{345}}$	$\underline{e_{125}}$	$\underline{-e_{1}}$	$\underline{-e_{235}}$	Ω	$\underline{e_{3}}$	$\underline{-e_{5}}$	$\underline{e_{134}}$	$\underline{-e_{123}}$	$\underline{-e_{145}}$	$\underline{-e_{31}}$	$\underline{e_{3145}}$	$\underline{e_{51}}$	$\underline{-e_{1253}}$	$\underline{-e_{53}}$	$\underline{e_{135}}$
e_{12}	e_{12}	$\underline{e_{2}}$	$\underline{-e_{1}}$	e_{123}	e_{124}	e_{125}	e_{1234}	$\underline{e_{51}}$	e_{1245}	$\underline{-e_{23}}$	$\underline{-e_{25}}$	$\underline{e_{14}}$	$\underline{1}$	e_{1253}	$\underline{e_{31}}$	$\underline{-e_{42}}$	$\underline{-e_{135}}$	$\underline{e_{245}}$	$\underline{e_{4}}$	Ω	$\underline{e_{235}}$	$\underline{-e_{134}}$	$\underline{-e_{145}}$	$\underline{e_{3}}$	$\underline{e_{234}}$	$\underline{e_{5}}$	$\underline{e_{34}}$	$\underline{e_{53}}$	$\underline{e_{45}}$	$\underline{-e_{2345}}$	$\underline{e_{3145}}$	$\underline{e_{345}}$
e_{53}	e_{53}	$-e_{135}$	$-e_{235}$	$\underline{-e_{5}}$	e_{345}	$\underline{e_{3}}$	$\underline{e_{45}}$	$\underline{e_{23}}$	$\underline{-e_{34}}$	$\underline{-e_{51}}$	$\underline{e_{31}}$	$-e_{2345}$	e_{1253}	$\underline{1}$	$\underline{-e_{25}}$	$-e_{3145}$	$\underline{-e_{2}}$	$\underline{-e_{134}}$	Ω	$\underline{e_{4}}$	$\underline{-e_{1}}$	$\underline{e_{245}}$	$\underline{-e_{234}}$	$\underline{-e_{125}}$	$\underline{e_{145}}$	$\underline{e_{123}}$	$\underline{e_{1245}}$	$\underline{e_{12}}$	$\underline{-e_{1234}}$	$\underline{-e_{14}}$	$\underline{-e_{42}}$	$\underline{e_{124}}$
e_{23}	e_{23}	e_{123}	$\underline{e_{3}}$	$\underline{-e_{2}}$	e_{234}	e_{235}	$\underline{e_{42}}$	$\underline{-e_{53}}$	e_{2345}	$\underline{e_{12}}$	e_{1253}	$\underline{-e_{34}}$	$\underline{-e_{31}}$	$\underline{e_{25}}$	$\underline{1}$	e_{1234}	$\underline{e_{5}}$	Ω	$\underline{e_{134}}$	$\underline{-e_{245}}$	$\underline{-e_{125}}$	$\underline{e_{4}}$	$\underline{e_{345}}$	$\underline{e_{1}}$	$\underline{-e_{124}}$	$\underline{e_{135}}$	$\underline{e_{14}}$	$\underline{e_{51}}$	$\underline{-e_{3145}}$	$\underline{e_{1245}}$	$\underline{e_{45}}$	$\underline{e_{145}}$
e_{14}	e_{14}	$\underline{e_{4}}$	$-e_{124}$	$-e_{134}$	$\underline{-e_{1}}$	e_{145}	$\underline{-e_{31}}$	$-e_{1245}$	$\underline{e_{51}}$	$\underline{e_{34}}$	$\underline{-e_{45}}$	$\underline{-e_{12}}$	$\underline{e_{42}}$	$-e_{3145}$	e_{1234}	$\underline{1}$	Ω	$\underline{e_{5}}$	$\underline{-e_{2}}$	$\underline{e_{135}}$	$\underline{-e_{345}}$	$\underline{-e_{123}}$	$\underline{e_{125}}$	$\underline{e_{234}}$	$\underline{-e_{3}}$	$\underline{-e_{245}}$	$\underline{e_{23}}$	$\underline{-e_{2345}}$	$\underline{-e_{25}}$	$\underline{-e_{53}}$	$\underline{e_{1253}}$	$\underline{e_{235}}$
e_{235}	e_{235}	e_{1253}	$-e_{53}$	$-e_{25}$	$-e_{2345}$	e_{23}	$\underline{-e_{245}}$	$\underline{-e_{3}}$	$\underline{e_{234}}$	$\underline{e_{125}}$	$\underline{-e_{123}}$	$\underline{-e_{345}}$	$\underline{e_{135}}$	$\underline{-e_{2}}$	$\underline{e_{5}}$	Ω	$\underline{1}$	e_{1234}	$-e_{3145}$	$\underline{e_{42}}$	$\underline{-e_{12}}$	$\underline{-e_{45}}$	$\underline{e_{34}}$	$\underline{e_{51}}$	$-e_{1245}$	$\underline{-e_{31}}$	$\underline{e_{145}}$	$-e_{1}$	$\underline{-e_{134}}$	$\underline{-e_{124}}$	e_{4}	$\underline{e_{14}}$
e_{145}	e_{145}	e_{45}	e_{1245}	$-e_{3145}$	e_{51}	e_{14}	$\underline{e_{135}}$	$\underline{-e_{124}}$	$\underline{e_{1}}$	$\underline{e_{345}}$	$\underline{e_{4}}$	$\underline{-e_{125}}$	$\underline{-e_{245}}$	$\underline{e_{134}}$	Ω	$\underline{e_{5}}$	e_{1234}	$\underline{1}$	$\underline{e_{25}}$	$\underline{-e_{31}}$	$\underline{-e_{34}}$	e_{1253}	$\underline{e_{12}}$	e_{2345}	$\underline{-e_{53}}$	$\underline{e_{42}}$	$\underline{e_{235}}$	$\underline{e_{234}}$	$-e_{2}$	e_{3}	$\underline{e_{123}}$	$\underline{e_{23}}$
e_{124}	e_{124}	$-e_{42}$	$-e_{14}$	$-e_{1234}$	e_{12}	e_{1245}	$\underline{e_{123}}$	$\underline{e_{145}}$	$\underline{-e_{125}}$	$\underline{-e_{234}}$	$\underline{e_{245}}$	$\underline{-e_{1}}$	$\underline{e_{4}}$	Ω	$\underline{-e_{134}}$	$\underline{-e_{2}}$	$-e_{3145}$	$\underline{-e_{25}}$	$\underline{1}$	e_{1253}	$-e_{2345}$	$\underline{e_{31}}$	$\underline{-e_{51}}$	$\underline{-e_{34}}$	$\underline{e_{23}}$	$\underline{e_{45}}$	e_{3}	$\underline{e_{345}}$	$-e_{5}$	$\underline{-e_{235}}$	$\underline{-e_{135}}$	$\underline{e_{53}}$
e_{345}	e_{345}	e_{3145}	$-e_{2345}$	e_{45}	e_{53}	e_{34}	$\underline{e_{5}}$	$\underline{e_{234}}$	$\underline{e_{3}}$	$\underline{-e_{145}}$	$\underline{-e_{134}}$	$\underline{e_{235}}$	Ω	$\underline{e_{4}}$	$\underline{e_{245}}$	$\underline{-e_{135}}$	$\underline{-e_{42}}$	$\underline{e_{31}}$	e_{1253}	$\underline{1}$	$\underline{e_{14}}$	$\underline{-e_{25}}$	$\underline{-e_{23}}$	e_{1245}	$\underline{e_{51}}$	e_{1234}	$\underline{e_{125}}$	$\underline{e_{124}}$	$\underline{e_{123}}$	$-e_{1}$	e_{2}	$\underline{e_{12}}$
e_{135}	e_{135}	$-e_{53}$	$-e_{1253}$	e_{51}	e_{3145}	$-e_{31}$	$\underline{-e_{145}}$	$\underline{-e_{123}}$	$\underline{e_{134}}$	$\underline{-e_{5}}$	$\underline{e_{3}}$	Ω	$\underline{-e_{235}}$	$\underline{-e_{1}}$	$\underline{e_{125}}$	$\underline{e_{345}}$	$\underline{e_{12}}$	$\underline{e_{34}}$	$-e_{2345}$	$\underline{-e_{14}}$	$\underline{1}$	e_{1245}	$-e_{1234}$	$\underline{e_{25}}$	$\underline{-e_{45}}$	$\underline{-e_{23}}$	$\underline{-e_{245}}$	e_{2}	$\underline{e_{234}}$	$\underline{-e_{4}}$	$\underline{-e_{124}}$	$\underline{e_{42}}$
e_{234}	e_{234}	$-e_{1234}$	e_{34}	e_{42}	e_{23}	e_{2345}	$\underline{e_{2}}$	$\underline{-e_{345}}$	$\underline{-e_{235}}$	$\underline{e_{124}}$	Ω	$\underline{e_{3}}$	$\underline{e_{134}}$	$\underline{-e_{245}}$	$\underline{e_{4}}$	$\underline{e_{123}}$	$\underline{e_{45}}$	e_{1253}	$\underline{-e_{31}}$	$\underline{e_{25}}$	e_{1245}	$\underline{1}$	$\underline{e_{53}}$	$\underline{-e_{14}}$	$\underline{-e_{12}}$	e_{3145}	e_{1}	$\underline{e_{145}}$	$\underline{e_{135}}$	$\underline{e_{125}}$	$-e_{5}$	$\underline{e_{51}}$
e_{245}	e_{245}	$-e_{1245}$	e_{45}	e_{2345}	$-e_{25}$	$-e_{42}$	$\underline{e_{235}}$	$\underline{-e_{4}}$	$\underline{e_{2}}$	Ω	$\underline{-e_{124}}$	$\underline{-e_{5}}$	$\underline{e_{145}}$	$\underline{e_{234}}$	$\underline{-e_{345}}$	$\underline{-e_{125}}$	$\underline{-e_{34}}$	$\underline{-e_{12}}$	$\underline{e_{51}}$	$\underline{e_{23}}$	$-e_{1234}$	$\underline{-e_{53}}$	$\underline{1}$	e_{3145}	$-e_{1253}$	$\underline{e_{14}}$	$\underline{-e_{135}}$	$\underline{-e_{134}}$	e_{1}	$\underline{e_{123}}$	$\underline{-e_{3}}$	$\underline{e_{31}}$
e_{123}	e_{123}	e_{23}	e_{31}	e_{12}	e_{1234}	$-e_{1253}$	$\underline{-e_{124}}$	$\underline{e_{135}}$	Ω	$\underline{e_{2}}$	$\underline{e_{235}}$	$\underline{-e_{134}}$	$\underline{e_{3}}$	$\underline{e_{125}}$	$\underline{e_{1}}$	$\underline{-e_{234}}$	$\underline{-e_{51}}$	e_{2345}	$\underline{e_{34}}$	e_{1245}	$\underline{-e_{25}}$	$\underline{e_{14}}$	e_{3145}	$\underline{1}$	$\underline{e_{42}}$	$\underline{-e_{53}}$	$-e_{4}$	e_{5}	$\underline{e_{345}}$	$\underline{e_{245}}$	$\underline{e_{145}}$	$\underline{e_{45}}$
e_{134}	e_{134}	e_{34}	e_{1234}	$-e_{14}$	$-e_{31}$	$-e_{3145}$	$\underline{e_{1}}$	Ω	$\underline{-e_{135}}$	$\underline{-e_{4}}$	$\underline{e_{345}}$	$\underline{e_{123}}$	$\underline{-e_{234}}$	$\underline{-e_{145}}$	$\underline{e_{124}}$	$\underline{-e_{3}}$	$-e_{1245}$	$\underline{e_{53}}$	$\underline{-e_{23}}$	$\underline{-e_{51}}$	$\underline{e_{45}}$	$\underline{e_{12}}$	$-e_{1253}$	$\underline{-e_{42}}$	$\underline{1}$	e_{2345}	$-e_{2}$	$\underline{-e_{245}}$	$\underline{-e_{235}}$	e_{5}	$\underline{e_{125}}$	$\underline{e_{25}}$
e_{125}	e_{125}	e_{25}	e_{51}	e_{1253}	$-e_{1245}$	e_{12}	Ω	$\underline{e_{1}}$	$\underline{e_{124}}$	$\underline{-e_{235}}$	$\underline{e_{2}}$	$\underline{e_{145}}$	$\underline{e_{5}}$	$\underline{-e_{123}}$	$\underline{-e_{135}}$	$\underline{e_{245}}$	$\underline{e_{31}}$	$\underline{-e_{42}}$	$\underline{-e_{45}}$	e_{1234}	$\underline{e_{23}}$	e_{3145}	$\underline{-e_{14}}$	$\underline{e_{53}}$	e_{2345}	$\underline{1}$	$\underline{e_{345}}$	$-e_{3}$	e_{4}	$\underline{e_{234}}$	$\underline{e_{134}}$	$\underline{e_{34}}$
e_{1234}	e_{1234}	$\underline{e_{234}}$	$\underline{-e_{134}}$	$\underline{e_{124}}$	$\underline{-e_{123}}$	Ω	$\underline{e_{12}}$	$\underline{e_{3145}}$	$\underline{e_{1253}}$	$\underline{-e_{42}}$	$\underline{-e_{2345}}$	$\underline{-e_{31}}$	$\underline{e_{34}}$	$\underline{-e_{1245}}$	$\underline{e_{14}}$	$\underline{e_{23}}$	$\underline{e_{145}}$	$\underline{e_{235}}$	$-e_{3}$	$\underline{e_{125}}$	$\underline{-e_{245}}$	$-e_{1}$	$\underline{-e_{135}}$	e_{4}	e_{2}	$\underline{e_{345}}$	1	$-e_{45}$	e_{53}	$-e_{25}$	e_{51}	e_{5}
e_{1253}	e_{1253}	$\underline{-e_{235}}$	$\underline{e_{135}}$	$\underline{-e_{125}}$	Ω	$\underline{e_{123}}$	e_{1245}	$\underline{-e_{31}}$	$\underline{-e_{1234}}$	$\underline{-e_{25}}$	$\underline{e_{23}}$	$\underline{-e_{3145}}$	$\underline{e_{53}}$	$\underline{e_{12}}$	$\underline{e_{51}}$	$\underline{e_{2345}}$	e_{1}	$\underline{e_{234}}$	$\underline{e_{345}}$	$\underline{e_{124}}$	$-e_{2}$	$\underline{e_{145}}$	$\underline{-e_{134}}$	$-e_{5}$	$\underline{-e_{245}}$	e_{3}	e_{45}	1	$-e_{34}$	e_{42}	$-e_{14}$	e_{4}
e_{1245}	e_{1245}	$\underline{e_{245}}$	$\underline{-e_{145}}$	Ω	$\underline{e_{125}}$	$\underline{-e_{124}}$	$\underline{-e_{1253}}$	$\underline{-e_{14}}$	$\underline{e_{12}}$	$\underline{-e_{2345}}$	$\underline{e_{42}}$	$\underline{e_{51}}$	$\underline{e_{45}}$	$\underline{e_{1234}}$	e_{3145}	$\underline{-e_{25}}$	$\underline{-e_{134}}$	e_{2}	e_{5}	$\underline{e_{123}}$	$\underline{e_{234}}$	$\underline{e_{135}}$	$-e_{1}$	$\underline{e_{345}}$	$\underline{-e_{235}}$	$-e_{4}$	$-e_{53}$	e_{34}	1	$-e_{23}$	e_{31}	e_{3}
e_{3145}	e_{3145}	$\underline{-e_{345}}$	Ω	$\underline{e_{145}}$	$\underline{-e_{135}}$	$\underline{e_{134}}$	$\underline{e_{51}}$	$\underline{-e_{1234}}$	$\underline{e_{31}}$	$\underline{e_{45}}$	$\underline{e_{34}}$	$\underline{e_{1253}}$	$\underline{e_{2345}}$	$\underline{-e_{14}}$	$\underline{-e_{1245}}$	$\underline{-e_{53}}$	$\underline{-e_{124}}$	$-e_{3}$	$\underline{-e_{235}}$	e_{1}	e_{4}	$\underline{e_{125}}$	$\underline{e_{123}}$	$\underline{e_{245}}$	$-e_{5}$	$\underline{e_{234}}$	e_{25}	$-e_{42}$	e_{23}	1	$-e_{12}$	e_{2}
e_{2345}	e_{2345}	Ω	$\underline{e_{345}}$	$\underline{-e_{245}}$	$\underline{e_{235}}$	$\underline{-e_{234}}$	$\underline{e_{25}}$	$\underline{e_{34}}$	$\underline{e_{23}}$	$\underline{e_{1245}}$	$\underline{e_{1234}}$	$\underline{-e_{53}}$	$\underline{-e_{3145}}$	$\underline{-e_{42}}$	$\underline{e_{45}}$	$\underline{-e_{1253}}$	$-e_{4}$	$\underline{e_{123}}$	$\underline{-e_{135}}$	$-e_{2}$	$\underline{-e_{124}}$	e_{5}	e_{3}	$\underline{e_{145}}$	$\underline{e_{125}}$	$\underline{e_{134}}$	$-e_{51}$	e_{14}	$-e_{31}$	e_{12}	1	e_{1}
Ω	Ω	e_{2345}	e_{3145}	e_{1245}	e_{1253}	e_{1234}	$\underline{e_{125}}$	$\underline{e_{134}}$	$\underline{e_{123}}$	$\underline{e_{245}}$	$\underline{e_{234}}$	$\underline{e_{135}}$	$\underline{e_{345}}$	$\underline{e_{124}}$	$\underline{e_{145}}$	$\underline{e_{235}}$	$\underline{e_{14}}$	$\underline{e_{23}}$	$\underline{e_{53}}$	$\underline{e_{12}}$	$\underline{e_{42}}$	$\underline{e_{51}}$	$\underline{e_{31}}$	$\underline{e_{45}}$	$\underline{e_{25}}$	$\underline{e_{34}}$	e_{5}	e_{4}	e_{3}	e_{2}	e_{1}	1

3 The Clifford-Likeness of the Orientation Congruent Algebra

In this section we first derive a formula for the sign factor function σ that, by multiplying the Clifford product, converts it to the orientation congruent product. The sign factor function is significant for two reasons: first, it can be used theoretically to construct proofs; second, it can be used practically to compute the orientation congruent product by hand or through computer algebra systems.

We then establish the validity of the sign factor function formula for the orientation congruent product by proving that it satisfies the primed axioms given in the last section. Actually, we explicitly prove this for only the last three axioms in Axiom Set III′, III.4′, III.5′, and III.6′, in that these are the only primed axioms that are either material modifications of some unprimed axiom or are entirely new.

3.1 Sigma Orientation Congruent Product Definition by the Sign Factor Function

In this subsection we define the Clifford-like sigma orientation congruent algebra $\sigma\mathcal{OC}_{p,q}$ and provide formulas for computing it. For our purposes the term *Clifford-like*[30] shall mean that the orientation congruent product of two basis blades of $\mathcal{OC}_{p,q}$ or $\mathcal{C}\ell_{p,q}$ can be obtained by adding a *sign factor* $\sigma = \pm 1$ to their Clifford product. The following subsection demonstrates that the $\mathcal{OC}_{p,q}$ algebra of the primed GR axioms is a Clifford-like algebra. It accomplishes this by proving that the $\mathcal{OC}_{p,q}$ algebra is identical to (or, more properly, isomorphic with) the explicitly Clifford-like $\sigma\mathcal{OC}_{p,q}$ algebra.[31]

In accordance with this fundamental definition we give an explicit formula for σ as a function of the two basis blades in the product. From this first formula for σ as a function of two basis blades we then derive a formula for σ as a function of the grades of any two homogeneous multivectors, but parametrized by the grade of the t–vector part of their Clifford product. In the end, by using the fundamental decomposition of the Clifford product, we obtain an explicit expression for the orientation congruent product of two arbitrary multivectors in terms of the sign factor function and their Clifford product. The proof of the keystone algebra isomorphism Theorem 3.5 is delayed until the next subsection.

In this section general set-theoretic sets as well as sets of basis vectors[32] are written as upper case letters in a calligraphic font such as $\mathcal{A}$. However, the power set function $\mathscr{P}$ as well as sets of blades or general multivectors are written in

[30] See Lounesto ([33], pp. 284f) for a more a bit more abstract definition.

[31] In another view we are proving the *deductive equivalence* of the primed orientation congruent axioms of the last section with that section's unprimed Clifford algebra axioms but having added to them as an axiom of existence Def. 3.1 for the (sigma) orientation congruent product.

[32] We except from this rule the set of *all* basis vectors for an algebra which we also write in a script font as $\mathscr{B}$.

a script font. Also $\#(\mathcal{A})$ denotes the *cardinality* of a set $\mathcal{A}$; $\mathscr{P}(\mathcal{A})$, the *power set* of $\mathcal{A}$; $\mathcal{A}^c$, the *set complement* of $\mathcal{A}$; and $\pm\mathcal{A} \equiv \{\, a \mid \pm a \in \mathcal{A} \,\}$, the *negative extension* of $\mathcal{A}$. The symbol $\varnothing$ will stand for the *empty set* $\{\ \}$.

First, we define notations for an ordered, orthonormal, set of basis vectors and various sets of basis blades derived from it. Let $\mathscr{B} = \{\, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \,\}$ where $n = p + q$ be an ordered set[33] of mutually orthogonal unit basis vectors for $\mathcal{OC}_{p,q}$ and its corresponding $\mathcal{C}\ell_{p,q}$. Then $\mathscr{B}^\wedge$ will signify the set of basis blades for $\mathcal{OC}_{p,q}$ and $\mathcal{C}\ell_{p,q}$ generated from $\mathscr{B}$ by taking, for each subset of $\mathscr{B}$, the outer product[34] of all basis vectors in it in their prescribed order.[35] We use $\mathscr{B}^r$ to mean the set of basis blades generated by $\mathscr{B}$ which are of grade $2 \le r \le n$. We also make the definitions $\mathscr{B}^1 \equiv \mathscr{B}$ and $\mathscr{B}^0 \equiv 1$.

Next, we introduce a function *set* which is implicitly parametrized by some ordered, orthonormal, set of basis vectors $\mathscr{B}$ for the orientation congruent algebra $\mathcal{OC}_{p,q}$ and its corresponding Clifford algebra $\mathcal{C}\ell_{p,q}$. We define the *set* function $\mathrm{set}\colon \pm\mathscr{B}^\wedge \to \mathscr{P}(\mathscr{B})$ such that for any $A \in \pm\mathscr{B}^r$

$$\mathrm{set}(A) \equiv \begin{cases} \{\, \mathbf{e}_{i_j} \mid \mathbf{e}_{i_j} \in \mathscr{B} \text{ and } A = \pm\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \mathbf{e}_{i_j} \cdots \wedge \mathbf{e}_{i_r} \,\}, & \text{if } r \ge 2 \\ \{\, \mathbf{e}_i \mid \mathbf{e}_i \in \mathscr{B} \text{ and } A = \pm\mathbf{e}_i \,\}, & \text{if } r = 1 \\ \{\ \}, & \text{if } r = 0. \end{cases}$$

Therefore, in particular, $\mathrm{set}(\pm\mathbf{e}_i) = \{\, \mathbf{e}_i \,\}$ for $\mathbf{e}_i \in \mathscr{B}$, and $\mathrm{set}(\pm 1) = \varnothing$. We also extend the function set in the obvious way to $\mathrm{set}\colon \mathscr{P}(\pm\mathscr{B}^\wedge) \to \mathscr{P}(\mathscr{P}(\mathscr{B}))$, so that, in particular, $\mathrm{set}\colon \pm\mathscr{B}^\wedge \mapsto \mathscr{P}(\mathscr{B})$.

Definition 3.1
We define the sigma orientation congruent *algebra of a nondegenerate quadratic form $Q_{p,q}$, denoted by $\sigma\mathcal{OC}_{p,q}$, and with product denoted by a circled star*[36] $\circledast$, *as the real 2^n-dimensional Clifford-like algebra that is the multilinear extension to all multivectors of the multiplication rule*

$$\mathbf{e}_i \circledast \mathbf{e}_j = \sigma(\mathbf{e}_i, \mathbf{e}_j)\ \mathbf{e}_i \circ \mathbf{e}_j,^{37} \tag{3.1}$$

defined between all pairs of basis blades $\mathbf{e}_i, \mathbf{e}_j \in \mathscr{B}^\wedge$, where σ, the sign factor function of basis blades, $\sigma\colon \mathscr{B}^\wedge \times \mathscr{B}^\wedge \to \pm\{\, 1 \,\}$, *is defined such that for any* $\mathbf{e}_i, \mathbf{e}_j \in \mathscr{B}^\wedge$

$$\sigma(\mathbf{e}_i, \mathbf{e}_j) = (-1)^{\frac{1}{2}\#(\mathcal{A}\cap\mathcal{B})[2\#(\mathcal{A})\#(\mathcal{B}) + \#(\mathcal{A}\cap\mathcal{B}) + 1]}, \tag{3.2}$$

where we have let $\mathcal{A} = \mathrm{set}(\mathbf{e}_i)$ *and* $\mathcal{B} = \mathrm{set}(\mathbf{e}_j)$.

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[33] In this section $\mathscr{B}$ is *not* necessarily signature-ordered; that is, ordered such that all basis vectors of positive signature precede those of negative signature.

[34] Since the vectors in $\mathscr{B}$ are mutually orthogonal, $\mathbf{e}_i \wedge \mathbf{e}_j = \mathbf{e}_i \circ \mathbf{e}_j = \mathbf{e}_i \circledcirc \mathbf{e}_j$ for any $\mathbf{e}_i, \mathbf{e}_j \in \mathscr{B}$ (see [26], p. 15, eq. (89)). Therefore, $\mathscr{B}^\wedge = \mathscr{B}^\circ = \mathscr{B}^\circledcirc$ and any basis blade of $\mathcal{OC}_{p,q}$ is a basis blade of $\mathcal{C}\ell_{p,q}$. Also, we take the outer product of one factor to be that factor and the outer product of no factors to be the unit 1.

[35] Accordingly, the vectors in $\mathscr{B}$ are called the *generators* of $\mathcal{OC}_{p,q}$ (or $\mathcal{C}\ell_{p,q}$).

[36] Here we are using the symbol $\circledast$ for the product of the algebra $\sigma\mathcal{OC}_{p,q}$ at least until we prove that it is identical to the product $\circledcirc$ of the orientation congruent algebra $\mathcal{OC}_{p,q}$.

Now we begin to construct an explicit formula for the multilinear extension of eqs. (3.1) and (3.2) for the sigma orientation congruent product $\circledast$ of $\sigma\mathcal{OC}_{p,q}$ given in Def. 3.1 to arbitrary multivectors.

The next lemma provides a formula for the sign factor function in terms of the two basis blade factors and the resultant basis blade of their Clifford product based on the relationship between the Clifford product of two basis blades and the symmetric difference of the sets of basis vectors "in" each of them.

Lemma 3.2 *For any* $\mathbf{e}_i, \mathbf{e}_j \in \mathscr{B}^\wedge$, *if* $\mathcal{A} = \mathrm{set}(\mathbf{e}_i)$, $\mathcal{B} = \mathrm{set}(\mathbf{e}_j)$, *and* $\mathcal{C} = \mathrm{set}(\mathbf{e}_i \circ \mathbf{e}_j) = \mathrm{set}(\mathbf{e}_i \circledast \mathbf{e}_j) \in \mathrm{set}(\pm\mathscr{B}^\wedge)$, *we may write the sign factor function* σ *of Def. 3.1 as*

$$\sigma(\mathbf{e}_i, \mathbf{e}_j) = (-1)^{\frac{1}{8}[\#(\mathcal{A})+\#(\mathcal{B})-\#(\mathcal{C})][4\#(\mathcal{A})\#(\mathcal{B})+\#(\mathcal{A})+\#(\mathcal{B})-\#(\mathcal{C})+2]}. \tag{3.3}$$

Proof. As is well known, if Δ denotes the *symmetric difference* operator on sets, and $\setminus$ denotes the *set difference* operator, then for all finite sets $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{A} \cap \mathcal{B} = (\mathcal{A} \cup \mathcal{B}) \setminus (\mathcal{A} \,\Delta\, \mathcal{B}) \text{ and} \tag{3.4a}$$

$$\#(\mathcal{A} \cap \mathcal{B}) = \frac{1}{2}[\#(\mathcal{A}) + \#(\mathcal{B}) - \#(\mathcal{A} \,\Delta\, \mathcal{B})]. \tag{3.4b}$$

Also for any $\mathbf{e}_i, \mathbf{e}_j \in \mathscr{B}^\wedge$ we have

$$\mathrm{set}(\mathbf{e}_i) \,\Delta\, \mathrm{set}(\mathbf{e}_j) = \mathrm{set}(\mathbf{e}_i \circ \mathbf{e}_j) = \mathrm{set}(\mathbf{e}_i \circledast \mathbf{e}_j). \tag{3.5}$$

Then it is straightforward to rewrite eq. (3.2) of Def. 3.1 as eq. (3.3). ■

In all the above we have had $\mathrm{set}(\mathbf{e}_i \circ \mathbf{e}_j) = \mathrm{set}(\mathbf{e}_i \circledast \mathbf{e}_j) \in \mathrm{set}(\pm\mathscr{B}^\wedge) = \mathscr{P}(\mathscr{B})$[38] for any $\mathbf{e}_i, \mathbf{e}_j \in \mathscr{B}^\wedge$, thus ensuring that $\sigma(\mathbf{e}_i, \mathbf{e}_j)$ is well defined as a closed operation. Since generally the Clifford product of arbitrary (not necessarily basis) blades is no longer homogeneous, we may as well consider next the Clifford and orientation congruent products of homogeneous multivectors (which are not necessarily blades).

The form of eq. (3.3) for the sign factor function σ is suitable for generalization from products of basis blades to products of homogeneous multivectors. Simultaneously we parametrize σ by a grade index so that it is useful when $A \circ B$ is a general multivector rather than a blade in $\pm\mathscr{B}^\wedge$. With these changes in the definition of the sign factor function of *basis blades* $\sigma(\mathbf{e}_i, \mathbf{e}_j)$ in eq. (3.2) of Def. 3.1 we obtain the definition of the sign factor function of the *grades of homogeneous multivectors* $\sigma_t(r, s)$ in eq. (3.7) of the next theorem.

[37]Since $\mathbf{e}_i$ and $\mathbf{e}_j$ are basis blades, $\mathbf{e}_i \circ \mathbf{e}_j \in \pm\mathscr{B}^\wedge$. Stated more generally, this becomes $\pm\mathscr{B}^\wedge$ is closed under any of the exterior, Clifford, or (sigma) orientation congruent product's.

[38]Or, equivalently, $\mathrm{set}(\mathbf{e}_i \circ \mathbf{e}_j) = \mathrm{set}(\mathbf{e}_i \circledast \mathbf{e}_j) \subseteq \mathscr{B}$.

Theorem 3.3 *For any* homogeneous multivectors $A_r, B_s \in \sigma\mathcal{OC}_{p,q}$ *and* $\mathcal{C}\ell_{p,q}$ *the* multilinear extension *of the product* $\circledast$ *of the sigma orientation congruent algebra* $\sigma\mathcal{OC}_{p,q}$ *given by Def. 3.1 is*

$$A_r \circledast B_s = \sum_{t=|r-s|}^{r+s} \langle A_r \circledast B_s \rangle_t = \sum_{t=|r-s|}^{r+s} \sigma_t(r,s)\, \langle A_r \circ B_s \rangle_t, \tag{3.6}$$

where the sign factor function[39] $\sigma_t \colon \mathbb{Z}[0,n] \times \mathbb{Z}[0,n] \mapsto \pm\{\,1\,\}$*, now a function of the grades of* A_r, B_s *and parametrized by the grade* $t \in \mathbb{Z}[0,n]$ *of the* t*-vector part of* $A_r \circ B_s$*, is given by*

$$\sigma_t(r,s) = (-1)^{\frac{1}{8}[r+s-t][4rs+r+s-t+2]}. \tag{3.7}$$

Proof. The proof is immediate from Lem. 3.2 by the multilinearity of the Clifford product and the linearity of the grade selection operator. ■

Using eq. (3.6) to evaluate the right hand side of the next equation we finally obtain an expression for the *sigma orientation congruent product of multivectors in terms of the sign factor function* $\sigma_t(r,s)$ and the Clifford product.

Corollary 3.4 *For all* $A, B \in \sigma\mathcal{OC}_{p,q}$

$$A \circledast B = \sum_{r,s} \langle A \rangle_r \circledast \langle B \rangle_s \text{ as evaluted by eq. (3.6).} \tag{3.8}$$

Proof. The proof is immediate from Lem. 3.2 by the multilinearity of the Clifford product and the linearity of the grade selection operator. ■

3.2 Sigma Orientation Congruent Algebra Satisfaction of the GR Axioms

We next prove Thm. 3.5. This fundamental isomorphism theorem states that the orientation congruent product of Cor. 3.4, derived from the sign factor function and the fundamental decomposition of the Clifford product, and the orientation congruent product, defined by Axiom Sets I′, II′, and III′, are equivalent. Our theorem and proof closely follows a similar theorem of Lounesto and his proof ([33], pp. 282f).

In the following proof, as is allowed, we restrict the factors in all products to be basis blades in $\mathscr{B}^{\wedge}$. So from another viewpoint we are directly proving an implicit keystone theorem that the formula for the sign factor function given by eq. (3.2) in Def. 3.1 is correct. This equation is the foundation from which all of Lem. 3.2, Thm. 3.3, Cor. 3.4, and the Fundamental $\mathcal{OC}$ Product Decomposition Theorem (Thm. 4.2 of section 4) follow.

[39] Here we are about to use the convenient notation $\mathbb{Z}[a,b] \equiv \{\, i \mid i \in \mathbb{Z} \text{ and } a \le i \le b \,\}$.

Consider an arbitrary signature-ordered set of generators[40] for $\mathcal{OC}_{p,q}$ and $\mathcal{C}\ell_{p,q}$, $\mathscr{B} = \{\, \mathbf{e}_1, \ldots, \mathbf{e}_p, \mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+q} \,\}$, such that for all integers $1 \le i, j \le n$, where $n = p + q$,

$$\mathbf{e}_i{}^2 = \mathbf{e}_i \circledcirc \mathbf{e}_i = \mathbf{e}_i \circ \mathbf{e}_i = \begin{cases} +1, & \text{if} \quad\ \ 1 \le i \le p, \quad \text{and} \\ -1, & \text{if } p+1 \le i \le p+q = n, \text{ and} \end{cases} \tag{3.9}$$

$$\mathbf{e}_i \circledcirc \mathbf{e}_j = \mathbf{e}_i \circ \mathbf{e}_j = -\mathbf{e}_j \circledcirc \mathbf{e}_i = -\mathbf{e}_j \circ \mathbf{e}_i, \quad \text{if } i \ne j.$$

The Fundamental $\sigma\mathcal{OC}$-$\mathcal{OC}$ Algebra Isomorphism Theorem

Theorem 3.5 *The real 2^n-dimensional Clifford-like algebra $\sigma\mathcal{OC}_{p,q}$ that is the multilinear extension to all multivectors of the multiplication rule*

$$\mathbf{e}_i \circledast \mathbf{e}_j = \sigma(\mathbf{e}_i, \mathbf{e}_j)\; \mathbf{e}_i \circ \mathbf{e}_j,$$

between all pairs of basis blades $\mathbf{e}_i, \mathbf{e}_j \in \mathscr{B}^\wedge$, where σ is defined by eq. (3.2) of Def. 3.1, is identical to the orientation congruent algebra $\mathcal{OC}_{p,q}$ with product $\circledcirc$ defined by Axiom Sets I', II', and III'.

Proof. It is sufficient to show that $\sigma\mathcal{OC}_{p,q}$ is generated by n anticommuting vectors with squares of ± 1 given by eq. (3.9), has a unit element, and that the blades in $\mathscr{B}^\wedge$ satisfy Axioms III.4$'$, III.5$'$, and III.6$'$ under the $\circledast$ product of $\sigma\mathcal{OC}_{p,q}$.

Consider the first requirement. Since the product of $\sigma\mathcal{OC}_{p,q}$ is simply the Clifford product multiplied by a sign factor of ± 1, it has the same set of generators $\mathscr{B}$ as the Clifford algebra $\mathcal{C}\ell_{p,q}$. Thus this requirement is fulfilled.

Next consider the second requirement. By definition, for any $\mathbf{e}_i \in \mathscr{B}^\wedge$, $1 \circledast \mathbf{e}_i = \sigma(1, \mathbf{e}_i)\; 1 \circ \mathbf{e}_i$. But

$$\begin{aligned} \sigma(1, \mathbf{e}_i) &= (-1)^{\frac{1}{2}\#(\varnothing \cap \mathrm{set}(\mathbf{e}_i))[2\#(\varnothing)\#(\mathrm{set}(\mathbf{e}_i)) + \#(\varnothing \cap \mathrm{set}(\mathbf{e}_i)) + 1]} \\ &= (-1)^0 = 1, \end{aligned}$$

Inspection of eq. (3.2) shows that σ is symmetric in its arguments. Thus, $1 \circledast \mathbf{e}_i = 1 \circ \mathbf{e}_i = \mathbf{e}_i$ and both multiplications commutate. Therefore, as required, the unit of algebra $\sigma\mathcal{OC}_{p,q}$ exists; it is the scalar 1.

Proof for Axiom III.4$'$

We recall that Axiom III.4$'$ requires that the outer product of multivectors is associative:

For all $A, B, C \in \mathcal{OC}_{p,q}$ or $\mathcal{C}\ell_{p,q}$

$$(A \wedge B) \wedge C = A \wedge (B \wedge C).$$

[40] In other words, $\mathscr{B}$ is an arbitrary signature-ordered, orthonormal, set of basis vectors for V^n.

Restricting A, B, and C to be homogeneous we obtain this equation:

For all $A_r, B_s, C_t \in \mathcal{OC}_{p,q}$ or $\mathcal{C}\ell_{p,q}$

$$(A_r \wedge B_s) \wedge C_t = A_r \wedge (B_s \wedge C_t).$$

Subtituting the basis blades $\mathbf{e}_r \in \mathscr{B}^r$, $\mathbf{e}_s \in \mathscr{B}^s$, and $\mathbf{e}_t \in \mathscr{B}^t$ we have

$$(\mathbf{e}_r \wedge \mathbf{e}_s) \wedge \mathbf{e}_t = \mathbf{e}_r \wedge (\mathbf{e}_s \wedge \mathbf{e}_t). \tag{3.10}$$

Using eq. (2.2) of Def. 2.5 for the outer product we may write

$$\langle\langle \mathbf{e}_r \circledast \mathbf{e}_s \rangle_{r+s} \circledast \mathbf{e}_t \rangle_{r+s+t} = \langle \mathbf{e}_r \circledast \langle \mathbf{e}_s \circledast \mathbf{e}_t \rangle_{s+t} \rangle_{r+s+t}. \tag{3.11}$$

Applying eq. (3.1) of Def. 3.1 gives

$$\sigma(\mathbf{e}_r, \mathbf{e}_s)\, \sigma(\mathbf{e}_r \circ \mathbf{e}_s, \mathbf{e}_t)\, \langle\langle \mathbf{e}_r \circ \mathbf{e}_s \rangle_{r+s} \circ \mathbf{e}_t \rangle_{r+s+t} = \\ \sigma(\mathbf{e}_s, \mathbf{e}_t)\, \sigma(\mathbf{e}_s \circ \mathbf{e}_t, \mathbf{e}_r)\, \langle \mathbf{e}_r \circ \langle \mathbf{e}_s \circ \mathbf{e}_t \rangle_{s+t} \rangle_{r+s+t}. \tag{3.12}$$

Now we let $\mathcal{A} = \mathrm{set}(\mathbf{e}_r)$, $\mathcal{B} = \mathrm{set}(\mathbf{e}_s)$, and $\mathcal{C} = \mathrm{set}(\mathbf{e}_t)$ and use eq. (3.2) to perform the next two evaluations.

Evaluating the sign factor functions on the left hand side gives

$$\sigma(\mathbf{e}_r, \mathbf{e}_s)\, \sigma(\mathbf{e}_r \circ \mathbf{e}_s, \mathbf{e}_t) = (-1)^{\frac{1}{2}\#(\mathcal{A}\cap\mathcal{B})[2\#(\mathcal{A})\#(\mathcal{B})+\#(\mathcal{A}\cap\mathcal{B})+1]} \cdot \\ (-1)^{\frac{1}{2}\#((\mathcal{A}\,\Delta\,\mathcal{B})\cap\mathcal{C})[2\#(\mathcal{A}\,\Delta\,\mathcal{B})\#(\mathcal{C})+\#((\mathcal{A}\,\Delta\,\mathcal{B})\cap\mathcal{C})+1]}. \tag{3.13}$$

Evaluating the sign factor functions on the right hand side gives

$$\sigma(\mathbf{e}_s, \mathbf{e}_t)\, \sigma(\mathbf{e}_s \circ \mathbf{e}_t, \mathbf{e}_r) = (-1)^{\frac{1}{2}\#(\mathcal{B}\cap\mathcal{C})[2\#(\mathcal{B})\#(\mathcal{C})+\#(\mathcal{B}\cap\mathcal{C})+1]} \cdot \\ (-1)^{\frac{1}{2}\#((\mathcal{B}\,\Delta\,\mathcal{C})\cap\mathcal{A})[2\#(\mathcal{B}\,\Delta\,\mathcal{C})\#(\mathcal{A})+\#((\mathcal{B}\,\Delta\,\mathcal{C})\cap\mathcal{A})+1]}. \tag{3.14}$$

Using eqs. (3.4b) and (3.5) we observe that iff at least one of $\mathcal{A} \cap \mathcal{B}$, $\mathcal{A} \cap \mathcal{B}$, or $\mathcal{A} \cap \mathcal{B}$ is nonempty, both sides of eq. (3.12) are equal to 0. In this case the values of the sign factor functions are irrelevant.

If all of $\mathcal{A} \cap \mathcal{B}$, $\mathcal{B} \cap \mathcal{C}$, and $\mathcal{A} \cap \mathcal{C}$ are equal to $\varnothing$, both sides of eq. (3.12) are nonzero and the factor due to Clifford products on the left hand side of eq. (3.12) is equal to that on the right hand side.[41] In this case the question of equality in eq. (3.12) hinges only on the values of the sign factor functions.

Examining the right hand sides of both eqs. (3.13) and (3.14), we see that the first "cardinality factors," $\#(\mathcal{A} \cap \mathcal{B})$ and $\#(\mathcal{B} \cap \mathcal{C})$, in the exponent of the first -1 are obviously 0 when $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{C} = \mathcal{A} \cap \mathcal{C} = \varnothing$. Thus this first -1 raised to the power of zero becomes 1 in both eq. (3.13) and (3.14).

[41] For a proof see Ref. [26], p. 11, eq. 57.

Consider now the first cardinality factor of the second -1 on the right hand side of eq. (3.13); it is $\#((\mathcal{A}\,\Delta\,\mathcal{B})\cap\mathcal{C})$. We perform some elementary set-theoretic manipulations[42] on $(\mathcal{A}\,\Delta\,\mathcal{B})\cap\mathcal{C}$.

$$\begin{aligned}(\mathcal{A}\,\Delta\,\mathcal{B})\cap\mathcal{C} &= [(\mathcal{A}\cup\mathcal{B})\setminus(\mathcal{A}\cap\mathcal{B})]\cap\mathcal{C}\\ &= [(\mathcal{A}\cup\mathcal{B})\cap(\mathcal{A}\cap\mathcal{B})^c]\cap\mathcal{C}\\ &= [(\mathcal{A}\cup\mathcal{B})\cap\mathcal{C}]\cap(\mathcal{A}\cap\mathcal{B})^c\\ &= [(\mathcal{A}\cap\mathcal{C})\cup(\mathcal{B}\cap\mathcal{C})]\cap(\mathcal{A}\cap\mathcal{B})^c\end{aligned}$$

Since $\mathcal{A}\cap\mathcal{C}=\mathcal{B}\cap\mathcal{C}=\varnothing$ we find that $(\mathcal{A}\,\Delta\,\mathcal{B})\cap\mathcal{C}=\varnothing$. Therefore the first cardinality factor of the second -1 on the right hand side of eq. (3.13) is 0. This makes that exponentiated -1 become 1. Similar manipulations lead to the same conclusion for eq. (3.14). Thus the sign factor functions are all unity and we have proved the equality of both sides of eq. (3.12). This in turn implies that eq. (3.10) is true. Therefore Axiom III.4′ as restricted to the set of basis blades $\mathscr{B}^\wedge$ is satisfied under the $\circledast$ product of $\sigma\mathcal{OC}_{p,q}$.

PROOF FOR AXIOM III.5′

Axiom III.5′ requires that the square of an r-blade and the product of the quadratic forms of the vectors in it be equal. So we restrict formula (3.2) for the sign factor function σ to two identical basis blades $\mathbf{e}_i, \mathbf{e}_i \in \mathscr{B}$ and apply the set-theoretic identity $\mathcal{A}\cap\mathcal{A}=\mathcal{A}$ to get

$$\sigma(\mathbf{e}_i,\mathbf{e}_i) = (-1)^{\frac{1}{2}\#(\mathrm{set}(\mathbf{e}_i))[2\#(\mathrm{set}(\mathbf{e}_i))\#(\mathrm{set}(\mathbf{e}_i))+\#(\mathrm{set}(\mathbf{e}_i))+1]}.$$

Letting $\#(\mathrm{set}(\mathbf{e}_i)) = r$, we obtain

$$\sigma(\mathbf{e}_i,\mathbf{e}_i) = (-1)^{\frac{1}{2}r(2r^2+r+1)}.$$

Since $r(2r^2+r+1)\equiv r(r-1) \mod 4$, we have

$$\sigma(\mathbf{e}_i,\mathbf{e}_j) = (-1)^{\frac{1}{2}r(r-1)}.$$

Therefore, $\mathbf{e}_i\circledast\mathbf{e}_i = (-1)^{\frac{1}{2}r(r-1)}\mathbf{e}_i\circ\mathbf{e}_i$. However, $(-1)^{\frac{1}{2}r(r-1)}\mathbf{e}_i$ is just the usual formula for $\mathbf{e}_i{}^\dagger$ the reversion of $\mathbf{e}_i$. Thus, we obtain

$$\begin{aligned}\mathbf{e}_i\circledast\mathbf{e}_i &= \mathbf{e}_i{}^\dagger\circ\mathbf{e}_i\\ &= Q(\mathbf{e}_{j_1})\cdots Q(\mathbf{e}_{j_k})\cdots Q(\mathbf{e}_{j_r}).\end{aligned}$$

Here we have let $\mathbf{e}_i = \mathbf{e}_{j_1}\wedge\cdots\wedge\mathbf{e}_{j_k}\wedge\cdots\wedge\mathbf{e}_{j_r}$ with all $\mathbf{e}_{j_k}$ basis vectors. Hence, Axiom III.5′ as restricted to the set of basis blades $\mathscr{B}^\wedge$ is satisfied under the $\circledast$ product of $\sigma\mathcal{OC}_{p,q}$.

[42]We use the notation $\mathcal{A}^c$ for the *set complement* of $\mathcal{A}$.

Proof for Axiom III.6′

Axiom III.6′ requires that within $\mathcal{OC}_{p,q}$, or its arbitrary extension by one dimension, for all nonempty subsets $\mathscr{A}$ of multivectors there exists a (nonunique) nonscalar, unit magnitude blade $\boldsymbol{\omega}_{\mathscr{A}}$, called the counit of $\mathscr{A}$, which (a) commutes with all multivectors in $\mathscr{A}$ and (b) has the generalized commutativity of right $\boldsymbol{\omega}_{\mathscr{A}}$-complementation property for all multivectors in $\mathscr{A}$. In addition, Axiom III.6′ states that an extension of $\mathcal{OC}_{p,q}$ by one dimension always exists. So in the following proof the symbol $\mathscr{B}$ for the basis set will refer to either the original basis or its extension to $\mathscr{B} \cup \{\, \mathbf{e}_{n+1} \,\}$, if necessary.[43] Of course, the meaning of $\mathscr{B}^{\wedge}$ must also be modified to reflect any change made to that of $\mathscr{B}$.

As is sufficient for the proof we restrict $\mathscr{A}$ to be $\varnothing \subset \mathscr{A} \subseteq \mathscr{B}^{\wedge}$. Then we claim that any basis blade $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}$ such that $\#(\mathrm{set}(\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}))$ is odd and

$$\mathrm{set}(\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}) = \bigcup \mathrm{set}(\mathscr{A}) \cup \mathcal{B} \quad \text{for some } \mathcal{B} \subseteq \mathscr{B} \tag{3.15}$$

satisfies these requirements.

Preliminarily, our alleged counit $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}$ must be a nonscalar. Since we have required that $\#(\mathrm{set}(\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}))$ is odd, it must not be 0. So we cannot have $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} = \pm 1$.

To show that $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}$ has unit magnitude consider that we have defined it to be a basis blade. We have already shown that for an arbitrary $r \geq 1$ the $\sigma\mathcal{OC}_{p,q}$ algebra square of any basis r-blade $\mathbf{e}_i = \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_k} \wedge \cdots \wedge \mathbf{e}_{j_r}$ is given by $\mathbf{e}_i \circledast \mathbf{e}_i = Q(\mathbf{e}_{j_1}) \cdots Q(\mathbf{e}_{j_k}) \cdots Q(\mathbf{e}_{j_r})$. Since $Q(\mathbf{e}_{j_k}) = \pm 1$ for all $\mathbf{e}_{j_k}$, the product of all the $Q(\mathbf{e}_{j_k})$'s is ± 1. And since $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}$ is a nonscalar basis blade, its $\sigma\mathcal{OC}_{p,q}$ algebra square is ± 1. Hence its magnitude is 1.

For part (a) let $\mathbf{e}_k$ be a basis blade such that $\mathbf{e}_k \in \mathscr{A} \subseteq \mathscr{B}^{\wedge}$. Then

$$\begin{aligned} \mathbf{e}_k \circledast \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} &= \sigma(\mathbf{e}_k, \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}})\, \mathbf{e}_k \circ \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \quad \text{and} \\ \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \circledast \mathbf{e}_k &= \sigma(\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}, \mathbf{e}_k)\, \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \circ \mathbf{e}_k. \end{aligned}$$

As previously observed σ is patently commutative in its arguments. Therefore we must show that $\mathbf{e}_k \circ \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} = \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \circ \mathbf{e}_k$.

By eq. (3.15), $\varnothing \subset \mathrm{set}(\mathbf{e}_k) \subseteq \mathrm{set}(\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}})$. Therefore,

$$\begin{aligned} \mathbf{e}_k \circ \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} &= \mathbf{e}_k \cdot \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \\ \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \circ \mathbf{e}_k &= \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \cdot \mathbf{e}_k \end{aligned}$$

where the centered dot $\cdot$ denotes the *(Hestenes) inner product* of the Clifford algebra $\mathcal{C}\ell_{p,q}$. Next we apply the rule for commuting the inner product given by Harke (Ref. [26], eq. (22)) as

$$A_r \cdot B_s = (-1)^{s(s-r)} B_s \cdot A_r \quad \text{for } r \geq s.$$

With $A_r = \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}$ and $B_s = \mathbf{e}_k$ we see that

[43] Here the subscript $n+1$ is not intended to imply that $Q_(\mathbf{e}_{n+1})$ is necessarily negative and neither is the symbol Q meant to imply that $Q = Q_{p,q+1}$ rather than $Q = Q_{p+1,q}$.

$$\mathbf{e}_k \circ \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} = \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \circ \mathbf{e}_k,$$

since $r = \#(\mathrm{set}(\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}))$ is odd by assumption. Therefore, part (a) of Axiom III.6′ is satisfied.

Part (b) of Axiom III.6′ requires that the right $\boldsymbol{\omega}_{\mathscr{A}}$-complement is commutative among the two factors and the result of an orientation congruent product of multivectors:

For all $\mathscr{A}$ that are nonempty sets of multivectors, $\varnothing \subset \mathscr{A} \subseteq \mathcal{OC}_{p,q}$, all couints $\boldsymbol{\omega}_{\mathscr{A}}$ of $\mathscr{A}$, and all $A, B \in \mathscr{A}$

$$A^{\boldsymbol{\omega}_{\mathscr{A}}} \circledcirc B = A \circledcirc B^{\boldsymbol{\omega}_{\mathscr{A}}} = (A \circledcirc B)^{\boldsymbol{\omega}_{\mathscr{A}}}.$$

Subtituting the basis blades $\mathbf{e}_r, \mathbf{e}_r \in \mathcal{A}$, where $\mathbf{e}_r \in \mathscr{B}^r$ and $\mathbf{e}_s \in \mathscr{B}^s$, and $\mathbf{e}_t = \boldsymbol{\omega}_{\mathscr{A}} \in \mathscr{B}^t$ in the last equation, and subtituting the $\sigma\mathcal{OC}_{p,q}$ algebra product $\circledast$ for the orientation congruent algebra product $\circledcirc$, we obtain

$${\mathbf{e}_r}^{\mathbf{e}_t} \circledast \mathbf{e}_s = \mathbf{e}_r \circledast {\mathbf{e}_s}^{\mathbf{e}_t} = (\mathbf{e}_r \circledast \mathbf{e}_s)^{\mathbf{e}_t}.$$

By expanding and lowering the superscript $\mathbf{e}_t$ notation we arrive at

$$(\mathbf{e}_r \circledast \mathbf{e}_t) \circledast \mathbf{e}_s = \mathbf{e}_r \circledast (\mathbf{e}_s \circledast \mathbf{e}_t) = (\mathbf{e}_r \circledast \mathbf{e}_s) \circledast \mathbf{e}_t.$$

Applying eq. (3.1) of Def. 3.1 gives

$$\sigma(\mathbf{e}_r, \mathbf{e}_t)\, \sigma(\mathbf{e}_r \circ \mathbf{e}_t, \mathbf{e}_s)\, (\mathbf{e}_r \circ \mathbf{e}_t) \circ \mathbf{e}_s \tag{3.16a}$$
$$= \sigma(\mathbf{e}_s, \mathbf{e}_t)\, \sigma(\mathbf{e}_s \circ \mathbf{e}_t, \mathbf{e}_r)\;\; \mathbf{e}_r \circ (\mathbf{e}_s \circ \mathbf{e}_t) \tag{3.16b}$$
$$= \sigma(\mathbf{e}_r, \mathbf{e}_s)\, \sigma(\mathbf{e}_r \circ \mathbf{e}_s, \mathbf{e}_t)\, (\mathbf{e}_r \circ \mathbf{e}_s) \circ \mathbf{e}_t. \tag{3.16c}$$

The double Clifford products on the right side of all three expressions in eq. (3.16) are equal because the Clifford product is associative and because, as we have already proved, the couint $\mathbf{e}_t$ commutes with all multivectors in its "generating" set $\mathcal{A}$. So next we look at the sign factor functions in these expressions.

Let $\mathcal{A} = \mathrm{set}(\mathbf{e}_r)$, $\mathcal{B} = \mathrm{set}(\mathbf{e}_s)$, and $\mathcal{C} = \mathrm{set}(\mathbf{e}_t)$. Then starting with eq. (3.2) of Def. 3.1 we evaluate and simplify the sign factor functions of expressions (3.16a) and (3.16c), each in turn, until we obtain two equivalent expressions. The manipulation of expression (3.16b) analogously to what we do to (3.16a) is left to the curious reader.

Evaluating the sign factor functions of expression (3.16a) gives

$$\begin{aligned}\sigma(\mathbf{e}_r, \mathbf{e}_t)\, \sigma(\mathbf{e}_r \circ \mathbf{e}_t, \mathbf{e}_s) = {} & (-1)^{\frac{1}{2}\#(\mathcal{A}\cap\mathcal{C})[2\#(\mathcal{A})\#(\mathcal{C})+\#(\mathcal{A}\cap\mathcal{C})+1]} \\ & \cdot (-1)^{\frac{1}{2}\#((\mathcal{A}\,\Delta\,\mathcal{C})\cap\mathcal{B})[2\#(\mathcal{A}\,\Delta\,\mathcal{C})\#(\mathcal{B})+\#((\mathcal{A}\,\Delta\,\mathcal{C})\cap\mathcal{B})+1]}.\end{aligned}$$

Using set-theoretic identities[44] to simplify the above expression gives

$$\sigma(\mathbf{e}_r,\mathbf{e}_t)\,\sigma(\mathbf{e}_r\circ\mathbf{e}_t,\mathbf{e}_s) = (-1)^{\frac{1}{2}\#(\mathcal{A})[2\#(\mathcal{A})\#(\mathcal{C})+\#(\mathcal{A})+1]} \cdot (-1)^{\frac{1}{2}\#(\mathcal{B}\cap\mathcal{A}^c)[2\#(\mathcal{C}\cap\mathcal{A}^c)\#(\mathcal{B})+\#(\mathcal{B}\cap\mathcal{A}^c)+1]}.$$

Recall the identity $\#(\mathcal{B}\cap\mathcal{A}^c) = \#(\mathcal{B}) - \#(\mathcal{A}\cap\mathcal{B})$; also, since $\mathcal{A}\subseteq\mathcal{C}$, $\#(\mathcal{C}\cap\mathcal{A}^c) = \#(C) - \#(\mathcal{A})$. Substituting these in the last expression produces

$$\sigma(\mathbf{e}_r,\mathbf{e}_t)\,\sigma(\mathbf{e}_r\circ\mathbf{e}_t,\mathbf{e}_s) = (-1)^{\frac{1}{2}\#(\mathcal{A})[2\#(\mathcal{A})\#(\mathcal{C})+\#(\mathcal{A})+1]} \cdot (-1)^{\frac{1}{2}[\#(\mathcal{B})-\#(\mathcal{A}\cap\mathcal{B})]\cdot[2\{\#(C)-\#(\mathcal{A})\}\#(\mathcal{B})+\#(\mathcal{B})-\#(\mathcal{A}\cap\mathcal{B})+1]}.$$

Since $\#(\mathcal{C})$ must always be odd, we may further simplify to

$$\sigma(\mathbf{e}_r,\mathbf{e}_t)\,\sigma(\mathbf{e}_r\circ\mathbf{e}_t,\mathbf{e}_s) = (-1)^{\frac{1}{2}\#(\mathcal{A})[2\#(\mathcal{A})+\#(\mathcal{A})+1]} \cdot (-1)^{\frac{1}{2}[\#(\mathcal{B})-\#(\mathcal{A}\cap\mathcal{B})]\cdot[2\{1-\#(\mathcal{A})\}\#(\mathcal{B})+\#(\mathcal{B})-\#(\mathcal{A}\cap\mathcal{B})+1]}.$$

Multiplying out the exponents and simplifying them mod 2 gives

$$\sigma(\mathbf{e}_r,\mathbf{e}_t)\,\sigma(\mathbf{e}_r\circ\mathbf{e}_t,\mathbf{e}_s) = (-1)^{[\#(\mathcal{A})+\#(\mathcal{B})+\#(\mathcal{A})\#(\mathcal{B})+\#(\mathcal{A})\#(\mathcal{B})\#(\mathcal{A}\cap\mathcal{B})]} \cdot (-1)^{\frac{1}{2}\#(\mathcal{A})[\#(\mathcal{A})+1]} \cdot (-1)^{\frac{1}{2}\#(\mathcal{B})[\#(\mathcal{B})+1]} \cdot (-1)^{\frac{1}{2}\#(\mathcal{A}\cap\mathcal{B})[\#(\mathcal{A}\cap\mathcal{B})-1]}. \quad (3.17)$$

Now we shift attention to expression (3.16c) whose sign factor functions evaluate to give

$$\sigma(\mathbf{e}_r,\mathbf{e}_s)\,\sigma(\mathbf{e}_r\circ\mathbf{e}_s,\mathbf{e}_t) = (-1)^{\frac{1}{2}\#(\mathcal{A}\cap\mathcal{B})[2\#(\mathcal{A})\#(\mathcal{B})+\#(\mathcal{A}\cap\mathcal{B})+1]} \cdot (-1)^{\frac{1}{2}\#((\mathcal{A}\,\Delta\,\mathcal{B})\cap\mathcal{C})[2\#(\mathcal{A}\,\Delta\,\mathcal{B})\#(\mathcal{C})+\#((\mathcal{A}\,\Delta\,\mathcal{B})\cap\mathcal{C})+1]}.$$

Removing the odd factor $\#(\mathcal{C})$, multiplying out and separating certain exponents, replacing $\#((\mathcal{A}\,\Delta\,\mathcal{B})\cap\mathcal{C})$ with $\#(\mathcal{A}\,\Delta\,\mathcal{B})$, and applying mod 2 identities yields

$$\sigma(\mathbf{e}_r,\mathbf{e}_s)\,\sigma(\mathbf{e}_r\circ\mathbf{e}_s,\mathbf{e}_t) = (-1)^{[\#(\mathcal{A}\,\Delta\,\mathcal{B})+\#(\mathcal{A})\#(\mathcal{B})\#(\mathcal{A}\cap\mathcal{B})]} \cdot (-1)^{\frac{1}{2}\#(\mathcal{A}\cap\mathcal{B})[\#(\mathcal{A}\cap\mathcal{B})+1]} \cdot (-1)^{\frac{1}{2}\#(\mathcal{A}\,\Delta\,\mathcal{B})[\#(\mathcal{A}\,\Delta\,\mathcal{B})+1]}.$$

Replacing the symmetric difference operator according to the identity $\#(\mathcal{A}\,\Delta\,\mathcal{B}) = \#(\mathcal{A}) + \#(\mathcal{B}) - 2\#(\mathcal{A}\cap\mathcal{B})$ leads to

$$\sigma(\mathbf{e}_r,\mathbf{e}_s)\,\sigma(\mathbf{e}_r\circ\mathbf{e}_s,\mathbf{e}_t) = (-1)^{[\#(\mathcal{A})+\#(\mathcal{B})+\#(\mathcal{A})\#(\mathcal{B})\#(\mathcal{A}\cap\mathcal{B})]} \cdot (-1)^{\frac{1}{2}\#(\mathcal{A}\cap\mathcal{B})[\#(\mathcal{A}\cap\mathcal{B})+1]} \cdot (-1)^{\frac{1}{2}[\#(\mathcal{A})+\#(\mathcal{B})-2\#(\mathcal{A}\cap\mathcal{B})]\cdot[\#(\mathcal{A})+\#(\mathcal{B})-2\#(\mathcal{A}\cap\mathcal{B})+1]}.$$

Multiplying out exponential terms and simplifying produces

[44] We use the notation $\mathcal{A}^c$ for the *set complement* of $\mathcal{A}$.

$$\sigma(\mathbf{e}_r, \mathbf{e}_s)\, \sigma(\mathbf{e}_r \circ \mathbf{e}_s, \mathbf{e}_t) = (-1)^{[\#(\mathcal{A})+\#(\mathcal{B})+\#(\mathcal{A})\#(\mathcal{B})+\#(\mathcal{A}\cap\mathcal{B})+\#(\mathcal{A})\#(\mathcal{B})\#(\mathcal{A}\cap\mathcal{B})]} \cdot (-1)^{\frac{1}{2}\#(\mathcal{A})[\#(\mathcal{A})+1]} \cdot (-1)^{\frac{1}{2}\#(\mathcal{B})[\#(\mathcal{B})+1]} \cdot (-1)^{\frac{1}{2}\#(\mathcal{A}\cap\mathcal{B})[\#(\mathcal{A}\cap\mathcal{B})+1]}. \quad (3.18)$$

We leave to the reader the easy exercise of completing the proof by showing that eqs. (3.17) and (3.18) in these last forms are equivalent. Assuming this done, we have proved that, for the set of basis blades $\mathscr{B}^{\wedge}$, Axiom III.6′ is satisfied under the $\circledast$ product of the algebra $\sigma\mathcal{OC}_{p,q}$.

This completes the proof of Thm. 3.5 that the Clifford-like sigma orientation congruent algebra $\sigma\mathcal{OC}_{p,q}$ and its product $\circledast$ defined by the multilinear extension of eqs. (3.1) and (3.2) in Def. 3.1 are identical (isomorphic) to the orientation congruent algebra $\mathcal{OC}_{p,q}$ and its product $\circledcirc$ defined by the primed axioms of section 2. ■

Generally, from now on we will drop the word "sigma" to simply refer to the orientation congruent algebra and orientation congruent product, and substitute the symbols $\mathcal{OC}$ and $\circledcirc$ for $\sigma\mathcal{OC}$ and $\circledast$, respectively. However, to indicate that the orientation congruent product is being computed as the product of the sign factor function and the Clifford product we will refer to the orientation congruent product *in sigma form.*

4 Computer Software Implemetations of the Orientation Congruent Algebra

Here we give some algorithms for computing the $\mathcal{OC}$ product using existing computer software packages. The practical necessity of computer aided computation of Clifford algebra operations has been noted by at least one researcher.

> "Indeed, I have to admit my own frustration in not being able to do more than a line or two of computations without making a serious mistake. I believe that what is most needed in the area today is an efficient computer software package for carrying out symbolic calculations in geometric [Clifford] algebra." —Garret Sobczyk[45]

By converting the Clifford product to the orientation congruent product, the sign factor function provides a way to compute the later either automatically or manually. The algorithms exploiting this fact that we give here are as simple as possible within the limitations of the software packages used. Except for a few elementary remarks we will not investigate the efficiency of these methods.

Of the many possible computer software packages available we will discuss algorithms for just two prototypical examples: Mathematica and Clical.[46] Mathematica is adaptable to do Clifford algebra calculations through programming; on the other hand, Lounesto's MS-DOS program Clical is specifically designed to do them with built-in functions.

Of the four implementations discussed, only one does full-blown, basis-free symbolic manipulation of Clifford or $\mathcal{OC}$ algebra expressions. Although, of course, all the Mathematica ones could and the Clical one cannot. Nevertheless, all these implementations are useful within their limitations—even those which must express multivectors as linear combinations of basis blades.

An algorithm suited to Mathematica, which is a completely programmable, symbolic computer algebra system (CAS), will be different than one suited to Clical, which is a numerical software package that can only run scripts without loops or conditional branches. Also Clical is limited to dimensions $n \leq 10$. Consequently, in Mathematica, computation of the orientation congruent product may be done by straightforward translation of the fundamental decomposition in Thm. 4.2 below into a program of nested loops. While in Clical, the loops representing the fundamental decomposition must be *rolled out* into a sum of functions whose number and definition varies with the dimension of the base vector space V^n.

First, we derive the fundamental decomposition theorem of the orientation congruent product in sigma form; using it gives a basic efficiency improvement over an algorithm based on Thm. 3.3. Next, we present two Mathematica implementations based on the fundamental decomposition theorem as well as one that is fully symbolic and basis-optional. Last, we discuss the Clical implementation of the orientation congruent product as a sum of predefined functions.

[45] Ref. [41], p. 18.

[46] This software is available online from the sources in Ref. [30].

4.1 The Fundamental Decomposition Theorem of the Orientation Congruent Product

First we repeat Thm. 3.3 and Cor. 3.4 for easy reference.[47] Then we give the fundamental decomposition theorem for the Clifford product and derive the corresponding theorem for the orientation congruent product from it.

Theorem 3.3 *For any* homogeneous multivectors $A_r, B_s \in \sigma\mathcal{OC}_{p,q}$ *and* $\mathcal{C}\ell_{p,q}$ *the* multilinear extension *of the product* $\circledast$ *of the sigma orientation congruent algebra* $\sigma\mathcal{OC}_{p,q}$ *given by Def. 3.1 is*

$$A_r \circledast B_s = \sum_{t=|r-s|}^{r+s} \langle A_r \circledast B_s \rangle_t = \sum_{t=|r-s|}^{r+s} \sigma_t(r,s) \, \langle A_r \circ B_s \rangle_t, \tag{3.6}$$

where the sign factor function[39] $\sigma_t \colon \mathbb{Z}[0,n] \times \mathbb{Z}[0,n] \mapsto \pm\{\,1\,\}$, *now a function of the grades of* A_r, B_s *and parametrized by the grade* $t \in \mathbb{Z}[0,n]$ *of the* t*-vector part of* $A_r \circ B_s$, *is given by*

$$\sigma_t(r,s) = (-1)^{\frac{1}{8}[r+s-t][4rs+r+s-t+2]}. \tag{3.7}$$

Corollary 3.4 *For all* $A, B \in \sigma\mathcal{OC}_{p,q}$

$$A \circledast B = \sum_{r,s} \langle A \rangle_r \circledast \langle B \rangle_s \text{ as evaluted by eq. (3.6).} \tag{3.8}$$

The Fundamental Clifford Product Decomposition Theorem[48]

Theorem 4.1 *For all homogeneous multivectors* $A_r, B_s \in \mathcal{C}\ell_{p,q}$ *their Clifford product may be written as a sum of homogeneous multivectors*

$$\begin{aligned} A_r \circ B_s &= \langle A_r \circ B_s \rangle_{|r-s|} + \langle A_r \circ B_s \rangle_{|r-s|+2} + \cdots + \langle A_r \circ B_s \rangle_{r+s} \\ &= \sum_{k=0}^{m} \langle A_r \circ B_s \rangle_{|r-s|+2k}, \end{aligned} \tag{4.1}$$

where $m = \frac{1}{2}(\mathcal{D}_n(r+s) - |r-s|)$ *with index function*

$$\mathcal{D}_n(i) \equiv \begin{cases} i, & \text{if } 0 \le i \le n, \text{ and} \\ 2n - i, & \text{if } n \le i \le 2n. \end{cases} \tag{4.2}$$

[47] In reading this theorem and corollary please recall that after proving the algebra isomorphism Theorem 3.5 we have now dropped the word *sigma* to leave simply *orientation congruent* and substituted the symbols $\mathcal{OC}$ and $\circledcirc$ for $\sigma\mathcal{OC}$ and $\circledast$, respectively.

[48] A proof of the infinite n version of Thm. 4.1 is sketched by Hestenes and Sobczyk on p. 10 of Ref. [27]. Harke also mentions it in eq. (48) of Ref. [26]. This finite n form of the fundamental Clifford product decomposition is from Conradt Ref. [19], eqs. (16) and (17).

In the next theorem we display the result of inserting the above formula (3.7) for $\sigma_t(r,s)$ as a multiplier of the grade selected Clifford products in the fundamental decomposition of the Clifford product from eq. (4.1). Thm. 4.2 presents the *fundamental decomposition of the orientation congruent product in terms of the sign factor function* $\sigma_t(r,s)$ and the Clifford product (or, briefly, in *sigma form*).

The Fundamental $\mathcal{OC}$ Product Decomposition Theorem in sigma form

Theorem 4.2 *For all homogeneous multivectors $A_r, B_s \in \mathcal{OC}_{p,q}$ and $\mathcal{C}\ell_{p,q}$ their orientation congruent product may be written as a sum of homogeneous multivectors*

$$\begin{aligned} A_r \circledcirc B_s = {} & \sigma_{|r-s|}(r,s)\,\langle A_r \circ B_s\rangle_{|r-s|} \\ & + \sigma_{|r-s|+2}(r,s)\,\langle A_r \circ B_s\rangle_{|r-s|+2} + \cdots \\ & + \sigma_{r+s}(r,s)\,\langle A_r \circ B_s\rangle_{r+s} \\ = {} & \sum_{k=0}^{m} \sigma_{|r-s|+2k}(r,s)\,\langle A_r \circ B_s\rangle_{|r-s|+2k} \end{aligned} \tag{4.3}$$

where the summation limit m and the index function $\mathcal{D}_n(i)$ are the same as for eq. (4.1), and $\sigma_t(r,s)$ is given by eq. (3.7) in Thm. 3.3.

Proof. The proof is immediate from Lem. 3.2 and Thm. 4.1 by the multilinearity of the Clifford product and the linearity of the grade selection operator. ∎

The number of grade selections (and consequent orientation congruent product evalutions) is reduced by at least $\min(r,s)$ when eq. (4.3) from Thm. 4.2 above is employed instead of eq. (3.6). Using the index function defined in the theorem, $\mathcal{D}_n(i)$, to determine the upper summation limit, m, reduces the number of products computed even more than the lower bound of $\min(r,s)$.

The Mathematica function `OCpD` given below in Fig. 4.1 achieves this maximum efficiency. `OCpD` is defined in terms of functions from the package Clifford which does *not* require a dimension n to be declared. Therefore, the parameter n of $\mathcal{D}_n(i)$ is set equal to the highest index of any of basis vectors in A_r or B_s. Using this value for n has exactly the same effect on the computational efficiency of a fundamental decomposition based algorithm as would using a value that is the dimension of any base space V^n that allows both A_r and B_s to be nonzero.

However, this is possible only when multivectors are expressed as linear combinations of basis blades, as is done in the package Clifford. If the dimension n is not fixed or known, using a basis-free algorithm based on the fundamental decomposition extracts a penalty of inefficiency. Then we must fall back on the least efficient basis-free strategy, abandoning the index function and simply setting $m = r + s$. Still, in comparison with eq. (3.6), the number of evalutions of Clifford products is reduced by $\min(r,s)$ in absolute terms. In the limit of infinite $\min(r,s)$, the fractional reduction is one half.

4.2 $\mathcal{OC}$ Computations in Mathematica Using *Clifford*

The author has programmed eq. (4.3) in a Mathematica notebook as the external function `OCpD` of Fig. 4.1. She based this function and auxilliary ones (not given) on the existing package *Clifford* which she has slightly modified. This package is internally titled "Clifford Algebra of a Euclidean Space" by its authors Oscar G. Caballero and José Luis Aragón Vera.[49] It computes Clifford algebra and quaternion operations in terms of the basis blades constructed from an orthonormal set of basis vectors denoted by `e[1]`, `e[2]`, ..., `e[n]`.

```
(* ************************************************** *)
(*                  Define OCpD ver. 1                *)
(*   Orien. Cong. Product in Fund. Decomposition Form  *)
(* ************************************************** *)
ClearAll[OCpD]
Remove[OCpD]
OCpD[x_, y_] := Module[{xGradeMin, xGradeMax, yGradeMin, yGradeMax,
    xyDimMax, Dind, TempSum, r, s, k},
  xGradeMin = GradeMin[x];  xGradeMax = GradeMax[x];  yGradeMin = GradeMin[y];
  yGradeMax = GradeMax[y];  xyDimMax = Max[DimMax[x], DimMax[y]];
  Dind[i_Integer, n_Integer] :=
  Which[
    0 <= i && i <    n, i,
    n <= i && i <= 2 n, 2 n - i
  ];
  TempSum = 0;
  r = xGradeMin;
  While[r <= xGradeMax,
    s = yGradeMin;
    While[s <= yGradeMax,
      k = Abs[r-s];
      While[k <= Dind[r+s, xyDimMax],
        TempSum = TempSum + SFac[r, s, k] Grade[Gp[Grade[x, r], Grade[y, s]], k];
        k = k + 2
      ];
      s = s + 1
    ];
    r = r + 1
  ];
  TempSum
]
```

Figure 4.1: **External Mathematica Function `OCpD`.** This function gives the $\mathcal{OC}$ product based on the fundamental decomposition theorem, Thm. 4.2.

[49]This package is available online in two versions from the sources in Refs. [16] and [17].

The author has also programmed eq. (4.3) as a Mathematica function internally defined within an altered version of the Caballero and Aragón Vera package *Clifford*. This function `OCp` (Fig. 4.2 below) is a directly modified form of the the package's definition of `Gp`. It computes the $\mathcal{OC}$ product by a straightforward use of the sign factor function as a multiplier defined by the assignment `sff=(-1)^(gu (2 g1 g2 + gu + 1)/2)`. Since the loops needed to implement Thm. 4.2 are already built into the definition of `Gp`, the `OCp` function runs much more quickly than the external function `OCpD` of Fig. 4.1.

```
(* Begin OC Product Section *)
OCProduct[ _] := $Failed
OCProduct[m1_,m2_,m3__] := tmp[OCProduct[m1,m2],m3] /.
            tmp->OCProduct
OCProduct[m1_,m2_] := ocprod[Expand[m1],Expand[m2]] //
                     Expand
(* The next 3 assignments define the alias OCp. *)
OCp[ _] := $Failed
OCp[m1_,m2_,m3__] := tmp[OCp[m1,m2],m3] /.
            tmp->OCp
OCp[m1_,m2_] := ocprod[Expand[m1],Expand[m2]] //
                     Expand
ocprod[a_,y_] := a y /; FreeQ[a,e[_?Positive]]
ocprod[x_,a_] := a x /; FreeQ[a,e[_?Positive]]
ocprod[x_,y_] := Module[{
  p1=ntuple[x,Max[dimensions[x],dimensions[y]]],q=1,s,r={},r1={},
 p2=ntuple[y,Max[dimensions[x],dimensions[y]]],
 g1=grados[x],g2=grados[y],gu,sff},
 gu=p1.p2;
 sff=(-1)^(gu (2 g1 g2 + gu + 1)/2);
 s=Sum[p2[[m]]*p1[[n]],{m,Length[p1]-1},{n,m+1,Length[p2]}];
     r1=p1+p2;
       r=Mod[r1,2];
            Do[ If[r[[i]] == 1, q *= e[i]];
     If[r1[[i]] == 2, q *= bilinearform[e[i],e[i]]],{i,Length[r1]} ];
    (-1)^s*q*sff ]
ocprod[a_ x_,y_] := a ocprod[x,y] /; FreeQ[a,e[_?Positive]]
ocprod[x_,a_ y_] := a ocprod[x,y] /; FreeQ[a,e[_?Positive]]
ocprod[x_,y_Plus] := Distribute[tmp[x,y],Plus] /. tmp->ocprod
ocprod[x_Plus,y_] := Distribute[tmp[x,y],Plus] /. tmp->ocprod
(* End of OC Product Section *)
```

Figure 4.2: **Internal Mathematica Function `OCp`.** This defines the $\mathcal{OC}$ product as a modified version of the *Clifford* package's definition of the function `Gp`.

4.3 $\mathcal{OC}$ Computations in Mathematica Using *GrassmannAlgebra*

John Browne has developed the Mathematica package *GrassmannAlgebra* [9] to translate the many operations given by Hermann Grassmann in his original works on the calculus extension into a modern computer system. This powerful package provides a fully symbolic CAS that allows, but does not require, the use of a basis and that can accept general metrics.

At the author's request John Browne has derived the following function for the orientation congruent product [11]. It is based on the *generalized Grassmann product* $\underset{\lambda}{\Delta}$ of the package.

$$\underset{m}{\alpha} \underset{\lambda}{\circledcirc} \underset{k}{\beta} = \sum_{\lambda=0}^{\mathrm{Min}[m,k]} (-1)^{m\lambda(k+1)} \left(\underset{m}{\alpha} \underset{\lambda}{\Delta} \underset{k}{\beta} \right) \tag{4.4}$$

In Browne's package and book [10] the $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ above are called *elements* (of a multilinear space). This term refers to a general general multilinear object, but it implies that the object is not specifically given a geometric interpretation.

Using a general metric, Dr. Browne has also demonstrated the facility of his package for transforming the entries in the multiplication table of $\mathcal{OC}_3$ into expressions containing the exterior product and the various forms of inner product available in *GrassmannAlgebra* [11]. His presentation of these results in a Mathematica notebook required 35 pages to print onto letter size paper.

4.4 $\mathcal{OC}$ Computations in Clical

The orientation congruent product may also be calculated in Clical, although much less elegantly than in Mathematica, by rolling out the nested loops of a program based on its fundamental decomposition. Let $A_r, B_s \in \mathcal{OC}_{p,q}$ be blades. Then the fundamental decomposition theorem, Thm. 4.2, states that the product $A_r \circledcirc B_s$ is not necessarily homogeneous. This theorem is naturally parametrized by the pair of grades (r, s) of A_r and B_s.

However, the tables below, and the functions derived from them, are instead naturally parametrized by the dimension $n = p + q$ of the base vector space of a given Clifford algebra. This is because in Clical the dimension of the Clifford algebra $\mathcal{C}\ell_{p,q}$ in which one will calculate is fixed by first declaring its signature (p, q). Also the sign factor function σ is dependent on three grades: r, s, and t, where t is the grade of the t-vector part of the product, $\langle A_r \circledcirc B_s \rangle_t$.

Therefore, for Clical we define a sequence of *winnow* functions each of which is a sum of terms of the form $\sigma_t(r, s)\, \langle A_r \circ B_s \rangle_t = \langle A_r \circledcirc B_s \rangle_t$. The *order* of one of these functions is defined to be the lowest dimension that allows all its terms to be (potentially) nonzero. Then, the sum of these functions up to order n contains only the grade-selected parts of the product $\langle A_r \circledcirc B_s \rangle_t$ that are *just* permitted by the dimension n. Therefore, in general, the summands $\sigma_{|r-s|+2k}(r, s)\, \langle A_r \circ B_s \rangle_{|r-s|+2k}$ in the fundamental decomposition of $A_r \circledcirc B_s$ in sigma form for a given r and s appear in *several* winnow functions of *different* orders.

Let the parameter t represent the selected grade of an orientation congruent (or Clifford) product as in $\langle A_r \circledcirc B_s \rangle_t$. Then, quite generally,[50] we define the integer k to be one half the reduction of the grade of the product from the sum of the grades of its factors:

$$k \equiv \frac{1}{2}(r + s - t). \tag{4.5}$$

We may also express this relationship as

$$r + s = t + 2k. \tag{4.6}$$

As a guide to defining the sequence of winnow functions introduced above we construct tables, one for each integer $m \geq 0$, that display the values of t, k, $r + s$, and the pairs of grades of factors (r, s), *whose products* $\langle A_r \circ B_s \rangle_t$ *may first become nonzero when the dimension of the base vector space* n *is equal to* m. The rows in these tables are ordered from top to bottom by increasing t. We order the pairs of grades (r, s) in a row from left to right by increasing r. Of course, we also require that all values in these tables satisfy $t, k, r, s \in \mathbb{Z}[0, m]$.[51]

Four examples of these tables are given below as Tabs. 4.1, 4.2, 4.3, and 4.4 for m equal to 2, 3, 4, and 5, respectively. We will ignore the lining out of some terms; this, as well as the use of bold fonts, will be explained later.

t	k	$r+s$	(r,s)		
0	2	4	$\cancel{(2,2)}$		
1	1	3	$\cancel{(1,2)}$	$\cancel{(2,1)}$	
2	0	2	$\cancel{(0,2)}$	$\cancel{(1,1)}$	$\cancel{(2,0)}$

Table 4.1: The grades of factors and products that first may be nonzero when the dimension $n = m = 2$. The text explains the lined out pairs of factor grades.

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[50] Eq. (4.5) is seen to be the natural generalization of eq. (3.4b) with eq. (3.5) applied to it, if we let $A_r = \mathbf{e}_i$ and $B_s = \mathbf{e}_j$ with $\mathbf{e}_i, \mathbf{e}_j \in \mathscr{B}^\wedge$, and $\mathcal{A} = \mathrm{set}(\mathbf{e}_i)$, $\mathcal{B} = \mathrm{set}(\mathbf{e}_j)$, and $\mathcal{C} = \mathrm{set}(\mathbf{e}_i \circledcirc \mathbf{e}_j)$, and $k = \#(\mathcal{A} \cap \mathcal{B})$, $r = \#(\mathcal{A})$, $s = \#(\mathcal{B})$, and $t = \#(\mathcal{C})$.

[51] Here we have used the convenient notation $\mathbb{Z}[a, b] \equiv \{ i \mid i \in \mathbb{Z} \text{ and } a \leq i \leq b \}$.

t	k	$r+s$	(r,s)
0	3	6	$\cancel{(3,3)}$
1	2	5	$\cancel{(2,3)}$ $\cancel{(3,2)}$
2	1	4	$\cancel{(1,3)}$ $\boldsymbol{\cancel{(2,2)}}$ $\cancel{(3,1)}$
3	0	3	$\cancel{(0,3)}$ $\cancel{(1,2)}$ $\cancel{(2,1)}$ $\cancel{(3,0)}$

Table 4.2: The grades of factors and products that first may be nonzero when the dimension $n = m = 3$. The text explains the lined out pairs of factor grades.

t	k	$r+s$	(r,s)
0	4	8	$\cancel{(4,4)}$
1	3	7	$\cancel{(3,4)}$ $\cancel{(4,3)}$
2	2	6	$\cancel{(2,4)}$ $(3,3)$ $\cancel{(4,2)}$
3	1	5	$\cancel{(1,4)}$ $\boldsymbol{\cancel{(2,3)}}$ $\boldsymbol{\cancel{(3,2)}}$ $\cancel{(4,1)}$
4	0	4	$\cancel{(0,4)}$ $\cancel{(1,3)}$ $\cancel{(2,2)}$ $\cancel{(3,1)}$ $\cancel{(4,0)}$

Table 4.3: The grades of factors and products that first may be nonzero when the dimension $n = m = 4$. The text explains the lined out pairs of factor grades.

t	k	$r+s$	(r,s)
0	5	10	$\cancel{(5,5)}$
1	4	9	$\cancel{(4,5)}$ $\cancel{(5,4)}$
2	3	8	$\cancel{(3,5)}$ $(4,4)$ $\cancel{(5,3)}$
3	2	7	$\cancel{(2,5)}$ $(3,4)$ $(4,3)$ $\cancel{(5,2)}$
4	1	6	$\cancel{(1,5)}$ $\boldsymbol{\cancel{(2,4)}}$ $(3,3)$ $\boldsymbol{\cancel{(4,2)}}$ $\cancel{(5,1)}$
5	0	5	$\cancel{(0,5)}$ $\cancel{(1,4)}$ $\cancel{(2,3)}$ $\cancel{(3,2)}$ $\cancel{(4,1)}$ $\cancel{(5,0)}$

Table 4.4: The grades of factors and products that first may be nonzero when the dimension $n = m = 5$. The text explains the lined out pairs of factor grades.

The italicized clause occurring two paragraphs up may be put another way: for one of these tables m is the *minimum value* of the dimension n that permits any row to exist; that is, that allows all grade selected products $\langle A_r \circ B_s \rangle_t$ resulting from homogeneous factors with grades r and s given by all pairs displayed in a row to be, in general, nonzero. Then it is easily seen that for each row in the m-table

$$m = r + s - k. \tag{4.7a}$$

Applying eq. (4.6) we obtain

$$m = t + k.^{52} \tag{4.7b}$$

These two equations may be rearranged to also give

$$r + s = m + k \quad \text{and} \tag{4.8a}$$

$$t = m - k. \tag{4.8b}$$

Adding the last two equations and rearranging yields

$$t = 2m - (r + s). \tag{4.9}$$

We recognize the last equation as $t = \mathcal{D}_m(r + s)$ after applying the second line of the index function $\mathcal{D}_n(i)$ definition in eq. (4.2) from the fundamental decomposition theorem. This leads directly to the observation that each table is constructed so that $m \leq t \leq 2m$.

This is also why, in general, a given pair of factor grades tracks along a course of consecutive tables. Specifically, in agreement with the fundamental decomposition theorem, if the pair (r, s) occurs in position (i, j) in the m-table, it also appears in position $(i + 2, j + 1)$ in the $(m + 1)$-table, if $m + 1 \leq r + s \leq 2(m + 1)$. (Here we have anticipated the matrix interpretation of the next paragraph.)

The pairs of factor grades in each table may be indexed as $(r, s)_{i,j}$ so that they constitute a matrix[53] of ordered pairs in the last column of that table. Each row of $[(r, s)_{i,j}]$ is aligned with the corresponding values of the parameters t, k, and $r + s$.

The row and column indices of this matrix satisfy $i, j \in \mathbb{Z}[1, m + 1]$. The row index may be written in terms of the row parameter k by

$$i = m - k + 1. \tag{4.10}$$

Nonzero entries of each row of the matrix $[(r, s)_{i,j}]$ must satisfy $\min(r, s) \geq k$, in addition to the already derived $0 \leq r, s \leq m$ and $r + s = m + k$ with

[52] Equations (4.7a) and (4.7b) are the Clifford algebra analogues of the set-theoretic formulas $\#(\mathcal{A} \cup \mathcal{B}) = \#(\mathcal{A}) + \#(\mathcal{B}) - \#(\mathcal{A} \cap \mathcal{B})$, and $\#(\mathcal{A} \cup \mathcal{B}) = \#(\mathcal{A} \,\Delta\, \mathcal{B}) + \#(\mathcal{A} \cap \mathcal{B})$, respectively.

[53] Properly, of course, the objects containing these indexed pairs should be called *indexed tables* or *arrays,* since we are not defining matrix addition (let alone multiplication) for them. Also, we let missing entries in the table become doubly 0-valued entries $(0, 0)$ in the "matrix."

$r+s \in \mathbb{Z}[m, 2m]$. All matrix entries, including invalid ones that should be zero, are given in terms of the row parameter k and the column index j by

$$(r,s)_{i,j} = (k+j-1, m-j+1)_{i,j}. \tag{4.11}$$

Solving eq. (4.10) for k and substituting in the first half of the pair on the right hand side of eq. (4.11) gives

$$(r,s)_{i,j} = (m-i+j, m-j+1)_{i,j}. \tag{4.12}$$

Requiring that the first half of the pair on the right hand side of the last equation satisfies $0 \le r, s \le m$ yields

$$j \le i, \tag{4.13}$$

which expresses that the matrix $[(r,s)_{i,j}]$ is naturally *lower triangular.*

We now begin to define, as an example, a sequence of winnow functions whose sum is the orientation congruent product in an algebra of base dimension $m = n = p + q = 5$. These definitions are valid for all multivector arguments $A, B \in \mathcal{OC}_{p,q}$. We denote this product in the functional form $\mathtt{oc}(A,B)$ similar to the way it would appear in Clical. Clical provides the grade selection operator which we need. But, we write it in the usual way with angular brackets and a subscript indicating the grade r to be selected as $\langle A \rangle_r$ rather than as it would be written in Clical as $\mathtt{Pu}(r, A)$.

It is convenient to start by defining two *base* winnow functions that include terms that first become nonzero at a variety of dimensions. As such they are of *inhomogeneous* order and may be called simply *base functions.* The first of these base functions, $\mathtt{ocbaseone}(A,B)$, contains terms of lowest order zero; while the second, $\mathtt{ocbasetwo}(A,B)$, contains terms of lowest order three. The functions of homogeneous order start with $\mathtt{ocdimfour}(A,B)$ which as its name suggests is of order four.

For the definition of $\mathtt{ocbaseone}(A,B)$ we need the orientation congruent left and right *contraction operators,* $\rceil$ and $\ulcorner$, respectively. These may be defined by the following equations[54] written in terms of some operations and a constant[55] that all are available in Clical. Here both $\boldsymbol{I}$ and $\mathtt{j}$ represent the algebra pseudoscalar.

$$\begin{aligned} A \rceil B &= \boldsymbol{I}^{-1} \circ [(\boldsymbol{I} \circ B) \wedge A^{\dagger}] && \text{(in normal notation)} \\ \mathtt{oclcont(A,B)} &= \mathtt{j} \setminus ((\mathtt{j} * \mathtt{B}) \,\hat{}\, \mathtt{A}\tilde{}) && \text{(as in Clical)} \end{aligned} \tag{4.14}$$

$$\begin{aligned} A \ulcorner B &= [B^{\dagger} \wedge (A \circ \boldsymbol{I})] \circ \boldsymbol{I}^{-1} && \text{(in normal notation)} \\ \mathtt{ocrcont(A,B)} &= (\mathtt{B}\tilde{}\,\hat{}(\mathtt{A} * \mathtt{j}))/\mathtt{j} && \text{(as in Clical)} \end{aligned} \tag{4.15}$$

[54] These equations (4.14) and (4.15) are derived and proved valid in section 5. See Tab. 5.6, line (8).

[55] The constant `j` in these function definitions is predefined in Clical only for algebras $\mathcal{C}\ell_{p,q}$ of dimension $n = p + q \ge 3$. Clical predefines another constant `i` for $n \le 2$. The following Clical script defines a variable `jj` which is the pseudoscalar in any dimension Clical can handle, $0 \le n \le 10$ (semicolons are used here to indicate the end of a Clical script line): `jj = 0; jj = j; jj = jj + i;`

The first winnow function $\texttt{ocbaseone}(A, B)$ contains the (possibly null) terms that, for $A_r \circledcirc B_s$ with homogeneous operands, *are of extremum grade* $|r-s|$ *or* $r+s$ in the fundamental decomposition of the orientation congruent product. In other words, it contains all orientation congruent inner and outer product terms found in the orientation congruent product of general multivectors for any dimension n. In particular, the orientation congruent products for dimensions $n \le 2$ are completely contained in it.

The base winnow function $\texttt{ocbaseone}(A, B)$ is defined by[56]

$$\begin{aligned}\texttt{ocbaseone}(A, B) \equiv\ &+ A \urcorner B + A \ulcorner B - \frac{1}{2}\langle A \urcorner B + A \ulcorner B\rangle_0 \\ &+ (A - \langle A\rangle_0) \wedge (B - \langle B\rangle_0).\end{aligned} \tag{4.16}$$

The Clifford product *commutator* $\texttt{clcom}(A, B)$ is used in the definition the next winnow function. The definition of the commutator is valid for any dimension n and is given by

$$\texttt{clcom}(A, B) \equiv \frac{1}{2}(A \circ B - B \circ A). \tag{4.17}$$

The second winnow function $\texttt{ocbasetwo}(A, B)$ includes all terms of the orientation congruent product decomposition *that are not contained in the base winnow function* $\texttt{ocbaseone}(A, B)$ and that result from a product of factors at least one of which is of grade two. Accordingly, it may be nonzero only when $n \ge 3$. The commutator excludes all terms of orientation congruent products that are also orientation congruent inner or outer products; these are already included in $\texttt{ocbaseone}(A, B)$. The commutator neatly replaces grade selection for this purpose.

The base winnow function $\texttt{ocbasetwo}(A, B)$ is defined by

$$\begin{aligned}\texttt{ocbasetwo}(A, B) \equiv &- \texttt{clcom}(A, \langle B\rangle_2) - \texttt{clcom}(\langle A\rangle_2, B) \\ &+ \texttt{clcom}(\langle A\rangle_1, \langle B\rangle_2) + \texttt{clcom}(\langle A\rangle_2, \langle B\rangle_1) \\ &+ \texttt{clcom}(\langle A\rangle_2, \langle B\rangle_2),\end{aligned} \tag{4.18a}$$

or, equivalently,

$$\begin{aligned}\texttt{ocbasetwo}(A, B) \equiv\ &- \texttt{clcom}(A - \langle A\rangle_1 - \frac{1}{2}\langle A\rangle_2, \langle B\rangle_2) \\ &- \texttt{clcom}(\langle A\rangle_2, B - \langle B\rangle_1 - \frac{1}{2}\langle B\rangle_2).\end{aligned} \tag{4.18b}$$

We digress to explain the lined out pairs in $[(r, s)_{i,j}]$. These are simply the pairs of factors whose grade-selected product is either an inner product (in the first column or the main diagonal) or an outer product (in the last row),[57] together with those pairs of factors at least one of which is of grade

[56] Other expressions may serve as the definition of the function $\texttt{ocbaseone}(A, B)$.

[57] The $(r, s)_{m,1}$ and $(r, s)_{m,m}$ entries with a scalar part are pairs of factors whose orientation congruent product is at the same time both an orientation congruent inner and outer product.

two. In other words, these are all pairs of factors whose terms are included in the base winnow functions `ocbaseone`(A, B) or `ocbasetwo`(A, B). In addition to lining out, a bold font is used for the pairs of factors whose products are in `ocbasetwo`(A, B). The terms resulting from these lined out factor pairs must be excluded from the higher order functions we define next.

The statements above about which matrix entries are lined out may also be expressed algeraically in terms of r and s in the following complementary form. The factor pairs in $[(r, s)_{i,j}]$ that are *not* lined out must satisfy the additional conditions

$$r, s > 2, \tag{4.19a}$$

$$r, s < m, \text{ and} \tag{4.19b}$$

$$r + s > m. \tag{4.19c}$$

The winnow function of order 4 `ocdimfour`(A, B) holds all terms of the orientation congruent product decomposition *that are not contained in the base winnow functions* and that first may be nonzero when $n = 4$. It is defined by

$$\texttt{ocdimfour}(A, B) \equiv -\langle\langle A\rangle_3 \circ \langle B\rangle_3\rangle_2. \tag{4.20}$$

As an example calculation we find the sign in eq. (4.20) by evaluating the sign factor function in eq. (3.7), repeated here,

$$\sigma_t(r, s) = (-1)^{\frac{1}{8}[r+s-t][4rs+r+s-t+2]} \tag{3.7$'$}$$

with the values in eq. (4.20) above. This gives

$$\begin{aligned} &= (-1)^{\frac{1}{8}[3+3-2][4\cdot 3\cdot 3+3+3-2+2]} \\ &= (-1)^{\frac{1}{8}[4][42]} = (-1)^{21} = -1. \end{aligned}$$

The winnow function of order 5 `ocdimfive`(A, B) comprises all terms of the orientation congruent product decomposition *that are not contained in the base or lower order winnow functions* and that first may be nonzero when $n = 5$. It is defined by

$$\begin{aligned} \texttt{ocdimfive}(A, B) \equiv\ &+ \langle\langle A\rangle_3 \circ \langle B\rangle_3\rangle_4 \\ &- \langle\langle A\rangle_4 \circ \langle B\rangle_3\rangle_3 - \langle\langle A\rangle_3 \circ \langle B\rangle_4\rangle_3 \\ &+ \langle\langle A\rangle_4 \circ \langle B\rangle_4\rangle_2. \end{aligned} \tag{4.21}$$

Finally, summing all the above winnow functions (the base functions and the winnow functions of order $m \leq 5$) gives `oc`(A, B) which contains *all* terms of the orientation congruent product decomposition that could be nonzero when $n = 5$ (as well as some that could be nonzero when $n > 5$).

$$\begin{aligned} \texttt{oc}(A, B) \equiv\ &\texttt{ocbaseone}(A, B) + \texttt{ocbasetwo}(A, B) \\ &+ \texttt{ocdimfour}(A, B) + \texttt{ocdimfive}(A, B). \end{aligned} \tag{4.22}$$

We end this section by deriving a formula for T_m the number of terms in a winnow function of order $m \geq 4$. First, consider the number of terms, lined out or not, in a table of order m. Since there are $m+1$ rows in an m-order table and since it has a triangular shape this is just the sum of the first $m+1$ positive integers

$$S_{m+1} = \frac{1}{2}(m+1)(m).$$

If we remove a count of the pairs that give rise to the outer product, those in the last row of the table, and a count of the pairs that give rise to the inner product, those in the first column and on the main diagonal, we get the following formula for the sum of the first $m-1$ positive integers

$$S_{m-1} = \frac{1}{2}(m-1)(m-2).$$

Finally, we remove a count of the pairs with $r = 2$ or $s = 2$. Pairs with at least one 2 in either position always occur in the last three rows of a table of order $m \geq 4$. The pairs in the highest and lowest of these three rows are already excluded because the are in `ocbaseone`(A, B). In a table of order $m \geq 4$ the middle row always contains exactly two such pairs, neither of which are in `ocbaseone`(A, B). Thus, we must remove a count of two from S_{m-1}. Therefore, the formula for T_m, the number of terms in a winnow function of order $m \geq 4$, is given by

$$T_m = \frac{1}{2}(m-1)(m-2) - 2. \tag{4.23}$$

5 The Clifford and Orientation Congruent Contraction Operators

5.1 The Significance of the Contraction Operators

The contraction operators of Clifford algebra are of theoretical and practical significance. They are used theoretically, for example, by Fernández et al. ([25], p. 15) in an axiomatic exposition of the Clifford algebra $\mathcal{C}\ell_n$. Their approach exploits the *Cartan decomposition formula* for the Clifford product of a vector and multivector to deform the exterior algebra:

$$\mathbf{x} \circ u = \mathbf{x} \wedge u + \mathbf{x} \rfloor u \quad \text{for all } \mathbf{x} \in V \text{ and all } u \in \bigwedge V. \tag{5.1}$$

Here the lower-left hooked bar $\rfloor$ stands for the left *Clifford contraction operator.*

In the work of Fernández et al. as well as this paper the Cartan decomposition formula is explicitly or implicitly the basis for a calculational method for the algebra considered.[58] We say "calculational method" because this approach does not give a genuine axiomatization of $\mathcal{OC}_{p,q}$. Since under a basis change it is does not respect the grading of the elements of the exterior algebra, this method falls short of an axiomatic definition ([20], p. 45). Therefore, our GR axioms for the Clifford and orientation congruent algebras of a nondegenerate quadratic form, strictly speaking, define only a calculational scheme for or representation of these algebras' products in terms of the exterior product and the algebras' contraction operators.

However, this type of axiomatic approach is related to the more fundamental one of Chevalley who embeds the Clifford algebra as a subalgebra of the associated exterior algebra's endomorphism algebra through the Chevalley-operator representation (which Chevalley [18] based on the Cartan decomposition formula). In either approach the contraction operators are crucial.

Lounesto ([33], pp. 288–90) discusses the contraction operators while constructing the linear isomorphism $\bigwedge V \to \mathcal{C}\ell(Q)$. On the practical side Lounsesto ([32], pp. 143f) points out the awkwardness of substituting the more symmetrical dot product of Hestenes et al ([27], p. 6) in constructing proofs. Also Dorst in Refs. [21], p. 10, and [22], p. 47, reiterates Lounesto's complaints as well as discusses the difficulties removed by using the contraction operators rather than the Hestenes dot product in designing computer algebra systems for Clifford algebra. For more motivational material see the references cited above as well as Lounesto [31].

Because of the importance of the contraction operators, we present them for both the Clifford and orientation congruent algebras. We give a parallel exposition so that comparison between the two tracks may aid the reader's understanding.

[58] Of course, for the $\mathcal{OC}$ algebra of this paper we would need to subtitute the left orientation congruent contraction operator for the Clifford one in eq. (5.1).

5.2 Fundamental Definitions of the Contraction Operators

Here we will give two definitions, based on Lounesto ([33], pp. 288–90), of the four contractions $\{\,\text{left}, \text{right}\,\} \times \{\,\mathcal{C}\ell, \mathcal{OC}\,\}$. See also Dorst ([21], p. 8) for another exposition of the first derivation, and Fauser ([24], pp. 23f) for another version of both. Let Tab. 5.1 define notations for the four contraction operators.

	Left	Right
$\mathcal{C}\ell$	$\rfloor$	$\lfloor$
$\mathcal{OC}$	$\rceil$	$\lceil$

Table 5.1: **Notations for the Four Contraction Operators.**

We assume that we have already made the extension from $V^n \times V^n$ to $\bigwedge V^n \times \bigwedge V^n$ of the bilinear form[59] associated with a general (not necessarily nondegenrate) quadratic form Q, perhaps, by means such as the references cited above employ. A pair of angular brackets $\langle\cdot,\cdot\rangle$ will denote both the original, nonextended bilinear form and its extension.

A general contraction operator may be fundamentally defined as the *dual* or *adjoint* of a modified exterior multiplication with respect to some *pairing*,[60] Depending on the modifications made to exterior product this definition produces a different contraction operator. The modifications required to produce a Clifford or orientation congruent, left or right, contraction operator involve only the reversion of some of the terms. The pairing required is the multilinear extension of the bilinear form associated with the quadratic form of the Clifford or orientation congruent algebra.

The equations in Tab. 5.2 give *duality definitions* of the four contraction operators using an extended bilinear form based on a nondegenerate Q on V^n.[61]

	Left	Right
$\mathcal{C}\ell$	$\langle u \rfloor v, w\rangle \equiv \langle v, u^\dagger \wedge w\rangle$	$\langle u \lfloor v, w\rangle \equiv \langle u, w \wedge v^\dagger\rangle$
$\mathcal{OC}$	$\langle u \rceil v, w\rangle \equiv \langle v, w \wedge u\rangle$	$\langle u \lceil v, w\rangle \equiv \langle u, v \wedge w\rangle$

Table 5.2: **Duality Definitions of the Four Contraction Operators.** These definitions are valid for all $u, v, w \in \bigwedge V^n$, and for all extended bilinear forms $\langle\cdot,\cdot\rangle$ derived from a nondegenerate quadratic form.

[59] This concept was introduced earlier by Def. 2.1 under the notation $B_Q(\cdot,\cdot)$.

[60] A pairing, or bilinear form over $\mathbb{R}$, is defined as a bilinear map $B\colon U \times V \to \mathbb{R}$ where U and V are vector spaces over $\mathbb{R}$ ([42], p. 58). See fn. 3 for a definition of bilinearity. An example of such a pairing is the *scalar product* of multivectors in a Clifford algebra.

[61] We remind the reader that, as mentioned in the paper's Introduction, the notations for the reversion and grade involution of a multivector A that we use are, respectively, the superscript dagger as $A^\dagger$, and the right hooked overline, as $\overline{A}\!\rceil$, or the superscript symbol derived from it, the upper right "corner," as $A^{\urcorner}$.

A set of three equations may be derived from the duality definition of each contraction operator; or, conversely, a set of these three equations may be used define the contraction operator that corresponds to it. These sets of three equations may be used to reduce an expression involving the contraction operators to another containing multivectors of lower grade than those in the original expression. Interestingly, these *reduction definitions* are more general than the duality ones; they allow the use of an extended bilinear form that is derived from a general, *possibly degenerate* quadratic form.

The first equation in the set of three is the same for all four operators as is shown in Tab. 5.3. The other two equations in the set vary by the operator according to Tabs. 5.4 and 5.5.

	Left	Right
$\mathcal{C}\ell$	$\mathbf{x} \rfloor \mathbf{y} \equiv \langle \mathbf{x}, \mathbf{y} \rangle$	$\mathbf{x} \lfloor \mathbf{y} \equiv \langle \mathbf{x}, \mathbf{y} \rangle$
$\mathcal{OC}$	$\mathbf{x} \rceil \mathbf{y} \equiv \langle \mathbf{x}, \mathbf{y} \rangle$	$\mathbf{x} \lceil \mathbf{y} \equiv \langle \mathbf{x}, \mathbf{y} \rangle$

Table 5.3: **Reduction Definitions of the Four Contraction Operators: Part 1.** These definitions are valid for all $\mathbf{x}, \mathbf{y} \in V^n$, and for all extended bilinear forms $\langle \cdot, \cdot \rangle$ derived from a general, possibly degenerate quadratic form.

	Left	Right
$\mathcal{C}\ell$	$\mathbf{x} \rfloor (u \wedge v) \equiv (\mathbf{x} \rfloor u) \wedge v + \overline{u} \wedge (\mathbf{x} \rfloor v)$	$(u \wedge v) \lfloor \mathbf{x} \equiv u \wedge (v \lfloor \mathbf{x}) + (u \lfloor \mathbf{x}) \wedge \overline{v}$
$\mathcal{OC}$	$\mathbf{x} \rceil (u \wedge v) \equiv u \wedge (\mathbf{x} \rceil v) + (\mathbf{x} \rceil u) \wedge \overline{v}$	$(u \wedge v) \lceil \mathbf{x} \equiv (u \lceil \mathbf{x}) \wedge v + \overline{u} \wedge (v \lceil \mathbf{x})$

Table 5.4: **Reduction Definitions of the Four Contraction Operators: Part 2.** These definitions are valid for all $\mathbf{x} \in V^n$, for all $u, v \in \bigwedge V^n$, and for all extended bilinear forms $\langle \cdot, \cdot \rangle$ derived from a general, possibly degenerate quadratic form.

	Left	Right
$\mathcal{C}\ell$	$(u \wedge v) \rfloor w \equiv u \rfloor (v \rfloor w)$	$w \lfloor (u \wedge v) \equiv (w \lfloor u) \lfloor v$
$\mathcal{OC}$	$(u \wedge v) \rceil w \equiv u \rceil (v \rfloor w)$	$w \lceil (u \wedge v) \equiv (w \lceil u) \lceil v$

Table 5.5: **Reduction Definitions of the Four Contraction Operators: Part 3.** These definitions are valid for all $u, v, w \in \bigwedge V^n$, and for all extended bilinear forms $\langle \cdot, \cdot \rangle$ derived from a general, possibly degenerate quadratic form.

5.3 Derived Expressions for the Contraction Operators

Lounesto ([33], pp. 38f) defines the *Hodge dual* (or *star*) operator, written as a star $*$, for $\mathcal{C}\ell_3$. This definition is immediately generalizable to to $\mathcal{C}\ell_{p,q}$ because it can be straightforwardly seen to be equivalent to the fourth equation on p. 166 of Burke [13] (but with multivectors substituted for differential forms). Therefore, we give the following general definition of the Hodge dual operator. For all $u, v \in \mathcal{C}\ell_{p,q}$ and for all extended bilinear forms $\langle \cdot, \cdot \rangle$ derived from a general, possibly degenerate quadratic form

$$u \wedge *\, v = v \wedge *\, u = \langle u, v \rangle \boldsymbol{I}. \tag{5.2}$$

Employing this definition and some other results to be added to a later version of this paper we may derive the equivalent expressions for the contraction operators given in Tab. 5.6.

	Left	Right	
	$u \rfloor v$	$u \lfloor v$	
	$= (v^\dagger \lfloor u^\dagger)^\dagger = (u \rceil v^\dagger)^\dagger = v \lceil u^\dagger$	$= (v^\dagger \rfloor u^\dagger)^\dagger = (u^\dagger \lceil v)^\dagger = v^\dagger \rceil u$	(1)
	$= [u \wedge (v \circ \boldsymbol{I})] \circ \boldsymbol{I}^{-1}$	$= \boldsymbol{I}^{-1} \circ [(\boldsymbol{I} \circ u) \wedge v]$	(2)
$\mathcal{C}\ell$	$= \{\boldsymbol{I}^{-1} \circ [(\boldsymbol{I} \circ v^\dagger) \wedge u^\dagger]\}^\dagger$	$= \{[v^\dagger \wedge (u^\dagger \circ \boldsymbol{I})] \circ \boldsymbol{I}^{-1}\}^\dagger$	(3)
	$= \{*^{-1}[u \wedge *(v^\dagger)]\}^\dagger$	$= \{*[*^{-1}(u^\dagger) \wedge v]\}^\dagger$	(4)
	$= *[(*^{-1} v) \wedge u^\dagger]$	$= *^{-1}[v^\dagger \wedge (*\, u)]$	(5)
	$u \rceil v$	$u \lceil v$	
	$= (v^\dagger \lceil u^\dagger)^\dagger = (u \rfloor v^\dagger)^\dagger = v \lfloor u^\dagger$	$= (v^\dagger \rceil u^\dagger)^\dagger = (u^\dagger \lfloor v)^\dagger = v^\dagger \rfloor u$	(6)
	$= \{[u \wedge (v^\dagger \circ \boldsymbol{I})] \circ \boldsymbol{I}^{-1}\}^\dagger$	$= \{\boldsymbol{I}^{-1} \circ [(\boldsymbol{I} \circ u^\dagger) \wedge v]\}^\dagger$	(7)
$\mathcal{OC}$	$= \boldsymbol{I}^{-1} \circ [(\boldsymbol{I} \circ v) \wedge u^\dagger]$	$= [v^\dagger \wedge (u \circ \boldsymbol{I})] \circ \boldsymbol{I}^{-1}$	(8)
	$= *^{-1}[u \wedge (*\, v)]$	$= *[(*^{-1} u) \wedge v]$	(9)
	$= \{*[*^{-1}(v^\dagger) \wedge u^\dagger]\}^\dagger$	$= \{*^{-1}[v^\dagger \wedge *(u^\dagger)]\}^\dagger$	(10)

Table 5.6: **Derived Expressions for the Four Contraction Operators.** These expressions are valid for all $u, v \in \bigwedge V^n$. The star $*$ represents the *Hodge dual* operator (see the text for details).

$\boxed{\text{TO BE DEVELOPED FURTHER}}$

6 Some Algebras, Graphs, and Theory

This section introduces the *product sequence graph* (PSG) using the nonassociative algebra $\mathcal{U}$ with four basis elements as a simple example. Next we represent the complete multiplication table of $\mathcal{OC}_3$ by its product sequence graph. Then we unfold this graph into its infinite three-dimensional version: the *product sequence lattice* (PSL) of $\mathcal{OC}_3$.

Next we simplify the PSG of some algebras with eight basis elements by letting the presence of some edges and arcs be implied and not providing a vertex for the unit. We also modify the rules for translating a path traced on the graph into a product equation. This gives us the *reduced* product sequence graph (RPSG) which is essentially a diagram of the Fano projective plane.

Using the RPSG and Albuquerque and Majid's work on maximally graded, nonassociative algebras defined on $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \mathbb{Z}_2$ we compare four Clifford-like algebras. These are the Clifford algebra $\mathcal{C}\ell_{0,3}$, the orientation congruent algebra $\mathcal{OC}_3$, the *Cayley* or *octonion* algebra $\mathbb{O}$, and the *modified octonion* algebra $\mathbb{O}_m$. We also investigate the four *antialgebra* counterparts to the above algebras which are, respectively, $\mathcal{C}\ell_3$, $\mathcal{OC}_{0,3}$, the *antioctonion* algebra $\overline{\mathbb{O}}$, and the *modified antioctonion* algebra $\overline{\mathbb{O}}_m$.

6.1 The Product Sequence Graph of a Small Algebra

The product sequence graph of an algebra, if it exists, encodes the algeba's multiplication table. The PSG is a mixed graph with labeled[62] edges and arcs. It employs the *Hamiltonian triangle*[63] construction.

The product sequence graph and its derivatives are called *mixed* graphs because they may contain both directed edges and undirected or bidirected arcs. The term "arc" will always be used to refer to an undirected or bidirected edge. For our purposes it is not necessary to distinguish between an undirected arc and a bidirected arc. We will freely represent either type as an arc having no arrowheads, or as an arc having an arrowhead at each end that points away from the arc.

As an introductory example consider the nonassociative algebra $\mathcal{U}$ with basis elements 1, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. These basis elements all have unit squares; the remaining products are shown in Tab. 6.1.

We derive the graph in Fig. 6.1 from the multiplication table of $\mathcal{U}$ by labeling its vertices with the basis elements of $\mathcal{U}$. The central purple vertex represents the unit 1. The other three vertices correspond to their labels $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. We claim it encodes the multiplication table of $\mathcal{U}$.

[62] Actually, as far as is practical and useful, we represent the labels of the edges and arcs by physically coloring them in our diagrams. But we do not use the adjective "edge-colored" to describe these graphs because it has long been widely adopted for a different concept.

[63] Porteous ([37], p. 183) has used this term in discussing the octonions. The Hamiltonian triangle is the graphical version of the familiar cyclic permutation rule for the vector cross product of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

		b			
	ab	1	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{c}$
a	1	1	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{c}$
	$\mathbf{a}$	$\mathbf{a}$	1	$\mathbf{c}$	$-\mathbf{b}$
	$\mathbf{b}$	$\mathbf{b}$	$-\mathbf{c}$	1	$\mathbf{a}$
	$\mathbf{c}$	$\mathbf{c}$	$\mathbf{b}$	$-\mathbf{a}$	1

Table 6.1: **The Multiplication Table for the Algebra $\mathcal{U}$.**

Examining the yellow triangle formed by the three directed edges connecting $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, we see that any sequence of vertex labels, v_1, v_2, v_3, written in the order encountered when tracing around the triangle in the direction of the arrows satisfies the equation $v_1 v_2 = v_3$. For example, $\mathbf{bc} = \mathbf{a}$. If we trace a sequence of two yellow edges in the direction contrary to their arrows we generate a sequence of vertex labels v_1, v_2, v_3 corresponding to the equation $v_1 v_2 = -v_3$. For example, $\mathbf{cb} = -\mathbf{a}$.

Consider the light gray loop incident with the 1 vertex. Starting at the 1 vertex and tracing around it twice in either direction produces the sequence of vertex labels $1, 1, 1$ corresponding to the equation $1(1) = 1$.

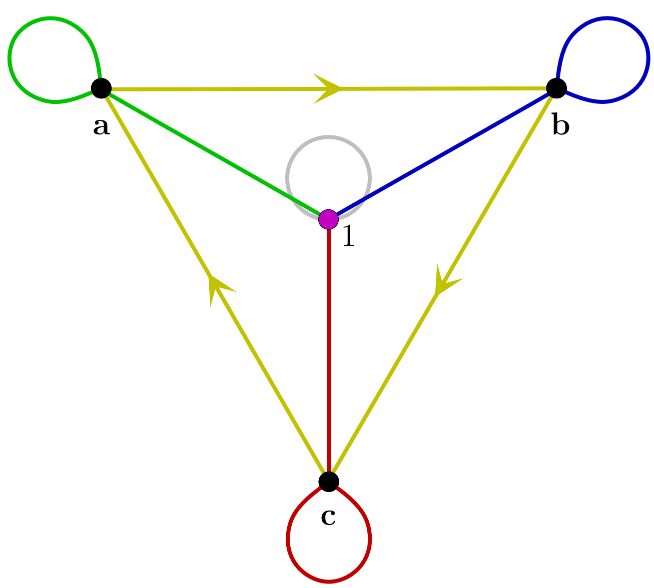


Figure 6.1: **The Product Sequence Graph of $\mathcal{U}$.**

Next examine the green "tennis racket" consisting of the undirected loop and the arc incident on the vertex labeled $\mathbf{a}$. Again tracing twice along these green arcs in any direction generates a sequence of three vertex labels v_1, v_2, v_3. Any one of these sequences satisfies the equation $v_1 v_2 = v_3$ if whenever you arrive at the $\mathbf{a}$ vertex having just traversed the loop you continue on the nonloop arc and vice versa. Tracing on the green arcs in this way generates the following equations all valid in $\mathcal{U}$: $1\mathbf{a} = \mathbf{a}$, $\mathbf{a}1 = \mathbf{a}$, and $\mathbf{aa} = 1$.

Tracing on each of the three tennis rackets following these rules gives $9 = 3\times 3$ equations; tracing on the light gray loop incident on the 1 vertex gives one equation; and, finally, tracing on the yellow triangle in all possible ways gives $6 = 2 \times 3$ equations. Our tracings have produced all $16 = 9 + 1 + 6$ equations corresponding to the 16 product entries in the multiplication table of $\mathcal{U}$. A graph for which this is possible is called the *product sequence graph* of an algebra.

Following these rules we may also chain together two or more tracings, each of length two, if the vertex at which the first one ends is also the vertex which begins the second and so on. Then, to form an equation from the sequence, every third vertex label in a three vertex sequence (except the final one) is replaced with the preceding two vertex labels enclosed in parenthesis. To complete the equation an equal sign is added just before the last vertex label. As before consecutive pairs of edges joining three vertices in a sequence must have the same color. For example, we may obtain the sequence $1, \mathbf{a}, \mathbf{a}, \mathbf{b}, \mathbf{c}$ which yields the equation $(1\mathbf{a})\mathbf{b} = \mathbf{c}$.

"Branching sequences" (corresponding to trees) may also be generated. Let us indicate each three element sequence in a side branch (as well as those in straight chains) by enclosing it within square brackets. For example, on the product sequence graph of $\mathcal{U}$ we might trace out the branching sequence $[1, \mathbf{a}, \mathbf{a}], [\mathbf{c}, \mathbf{a}, \mathbf{b}], \mathbf{c}$ which yields the equation $(1\mathbf{a})(\mathbf{ca}) = \mathbf{c}$. A more complicated example: the branching sequence $[1, [\mathbf{b}, \mathbf{c}, \mathbf{a}], \mathbf{a}], [\mathbf{c}, \mathbf{a}, \mathbf{b}], \mathbf{c}$ translates into the equation $(1(\mathbf{bc}))(\mathbf{ca}) = \mathbf{c}$.

The astute reader may have realized that $\mathcal{U}$ is isomorphic to any of four subalgebras of $\mathcal{OC}_3$. For example, applying the correspondences $1 \leftrightarrow 1$, $\mathbf{a} \leftrightarrow \mathbf{e}_1$, $\mathbf{b} \leftrightarrow \mathbf{e}_2$, $\mathbf{c} \leftrightarrow \mathbf{e}_{12}$ yields the imperfect orientation congruent algebra $\mathcal{IOC}_2$.

6.2 The Product Sequence Graph of $\mathcal{OC}_3$

Fig. 6.2 presents the product sequence graph of $\mathcal{OC}_3$. Here we have used as vertex labels the sequences of indices normally appearing in multi-index notation as subscripts to the base symbol "$\mathbf{e}$." However, for the unit 1 we have substituted "$\varnothing$" as the empty sequence. (The notation in Fig. 6.2 was changed only for the convenience of the author in constructing the diagram.)

Also in Fig. 6.2 to avoid drawing a messy diagram with close or crossing edges we have split the unit element's vertex into seven purple ones each labeled "$\varnothing$." We agree to interpret these seven as one "super vertex" so that having arrived at any one of them we may jump to any of the other six and continue on from it (assuming in doing so we are adhering to all other rules).

We have run out of easily distinguishable colors for the edges. However, in the sense of an abstract edge-labeled graph, the reader should consider the following sets of arcs to have dinstinct colors: each of the seven black tennis rackets; and each of the three sets of three closely-spaced orange arcs drawn in the shape of a circlar arc and incident on the central vertex labeled “123” or two diametrically opposite vertices.

Six vertices in Fig. 6.2 lie on a thin circular line. It represents no edge; but it does suggest a great circle of a sphere. The three sets of three closely-spaced orange arcs may appear to lie on other great circles. The reader may imagine the three-dimensional placement of the other elements of the graph.

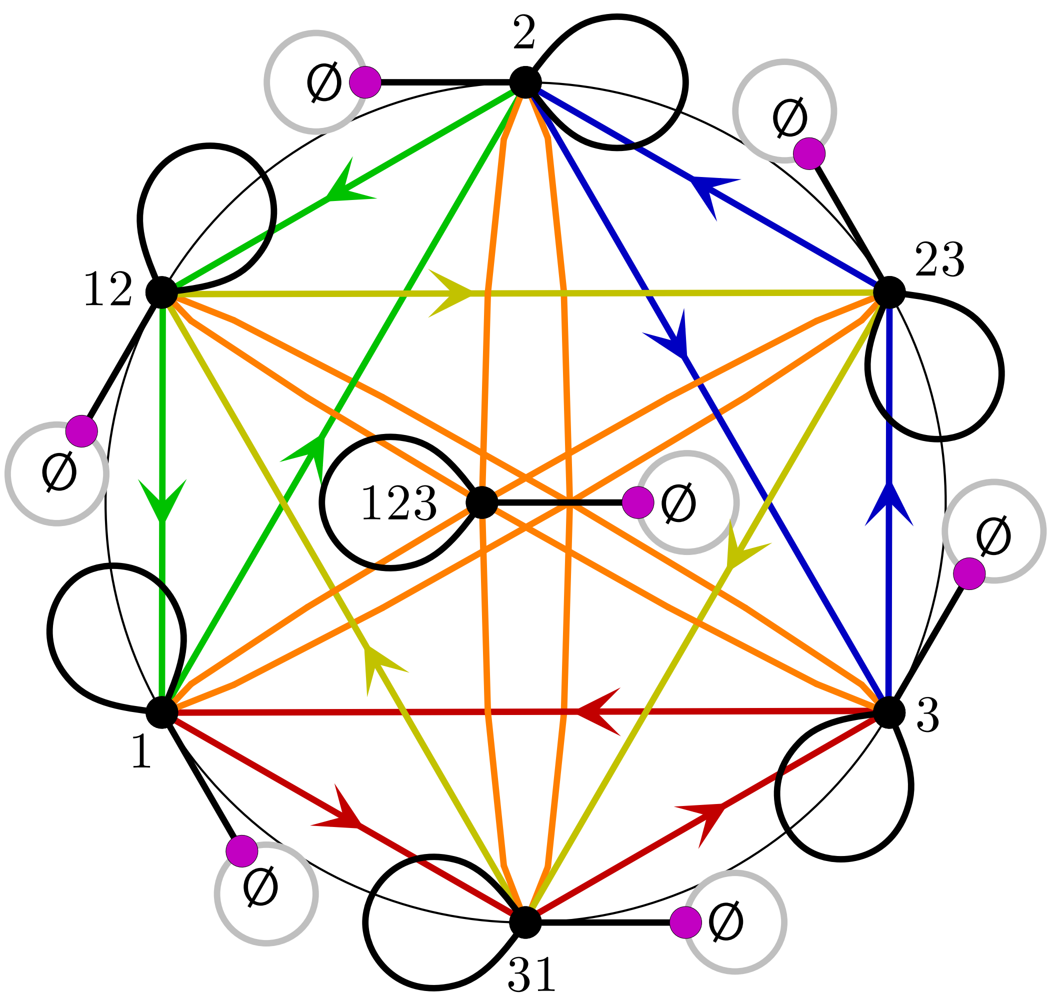


Figure 6.2: **The Product Sequence Graph of $\mathcal{OC}_3$.**

6.3 The Product Sequence Lattice of $\mathcal{OC}_3$

The product sequence graph of $\mathcal{OC}_3$ may be unfolded into an infinite three-dimensional lattice-like graph. Fig. 6.3 is a finite section of one plane in such a graph. As in Fig. 6.2 we have used the multi-index subscript sequences, but here, rather than labeling the vertices, these integer sequences directly represent them. Also in this figure an arc is indicated by a line segment having an arrowhead at each of its ends that points away from the segment.

The coloring scheme for the edges and arcs in this figure is different from that used in Fig. 6.2. Here moving along any two solid black edges in a straight

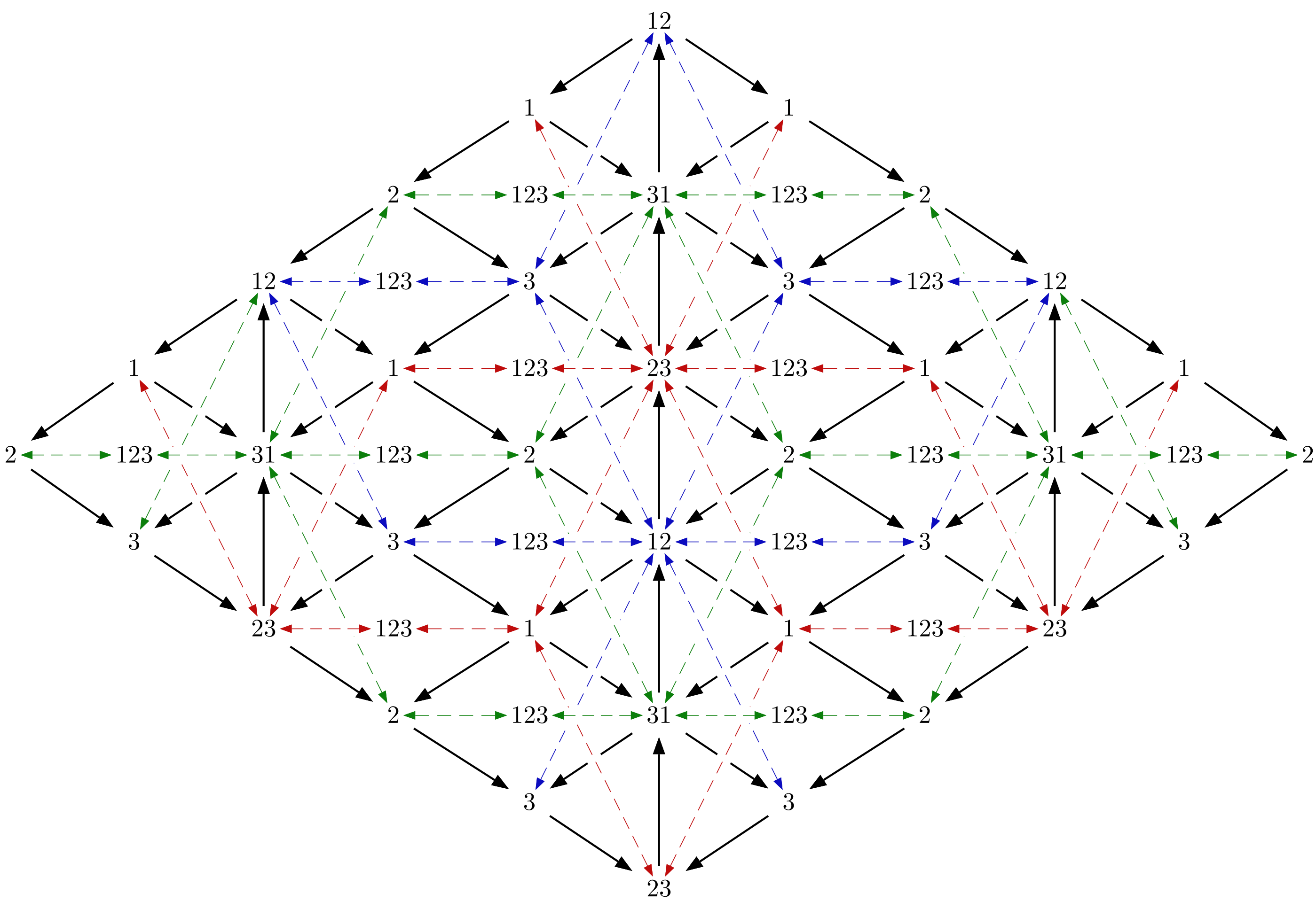


Figure 6.3: **The Product Sequence Lattice of $\mathcal{OC}_3$.** This is actually a finite section that tiles a plane. The text gives the spatial "tiling." These edge colors differ from those of Fig. 6.2.

path is considered remaining on edges of the same color. One may also move similarly along the thinner red, green, or blue dashed arcs in straight paths, but only if a "123" labeled vertex is incident between two adjacent arcs. After starting a two arc sequence from a vertex labeled "123" and reaching another vertex along a dashed red, green, or blue arc, one must continue on a dashed arc that is the same color and that makes the smallest angle with the just traversed arc.

This finite section (not of minimal size) will tile the plane to infinity. All of space may be "tiled" by infinitely repeating a three-dimensional unit comprising two alingned, stacked copies of this plane in turn stacked upon a third aligned plane of purple super vertices distributed in the same pattern as the vertices in the other two planes. These purple vertices represent the unit and so should be labeled "$\varnothing$." All nearby vertices in the third plane must be connected by straight, light gray arcs. The vertices immediately above or below each other along the stacking direction must be connected by appropriately colored arcs.

6.4 Eight Algebras in Graphs and Theory

In this subsection we provide the multiplication tables and reduced product sequence graphs of some Clifford, orientation congruent, and generalized Cayley algebras. Using their reduced product sequence graphs we directly compare four of these Clifford-like algebras. These are the Clifford algebra $\mathcal{C}\ell_{0,3}$, the orientation congruent algebra $\mathcal{OC}_3$, the *Cayley* or *octonion* algebra $\mathbb{O}$, and the *modified octonion* algebra $\mathbb{O}_m$. We also investigate the four *antialgebra* counterparts to the above algebras which are, respectively, $\overline{\mathcal{C}\ell_3}$, $\overline{\mathcal{OC}_{0,3}}$, the *antioctonion* algebra $\overline{\mathbb{O}}$, and the *modified antioctonion* algebra $\overline{\mathbb{O}}_m$.

However, these counterpart algebras can only be represented by reduced PSGs after they are *rectified* by multiplying their odd-grade basis blades by the imaginary unit i. The rectified forms of these algebras are isomorphic to their *opposite* algebras which in turn are isomorphic to the four original algebras.

Here we will also discuss those aspects of the work of Albuquerque and Majid (Refs. [3], [4], [5], and [6]) on the theory of maximally graded, nonassociative algebras defined on $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \mathbb{Z}_2$ and compare expressions for the Clifford, orientation congruent, and octonion algebra in their maximally graded forms.

TO BE DEVELOPED FURTHER

$a \circ b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$
$\mathbf{e}_1$	$\mathbf{e}_1$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_2$	$\mathbf{e}_3$	$\boldsymbol{I}$	$-\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	-1	$\mathbf{e}_{23}$	$\mathbf{e}_1$	$\boldsymbol{I}$	$-\mathbf{e}_3$	$-\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	-1	$\boldsymbol{I}$	$-\mathbf{e}_1$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$\boldsymbol{I}$	-1	$-\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$-\mathbf{e}_3$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$-\mathbf{e}_3$	$\boldsymbol{I}$	$\mathbf{e}_1$	$\mathbf{e}_{23}$	-1	$-\mathbf{e}_{12}$	$-\mathbf{e}_2$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_{31}$	$\mathbf{e}_{12}$	-1	$-\mathbf{e}_1$
$\boldsymbol{I}$	$\boldsymbol{I}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	1

(a) The Multiplication Table for the Clifford Algebra $\mathcal{C}\ell_{0,3}$.

$a \circledcirc b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_2$	$\mathbf{e}_3$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$\mathbf{e}_1$	$\boldsymbol{\Omega}$	$-\mathbf{e}_3$	$\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\boldsymbol{\Omega}$	$-\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$\boldsymbol{\Omega}$	1	$-\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$-\mathbf{e}_3$	$\boldsymbol{\Omega}$	$\mathbf{e}_1$	$\mathbf{e}_{23}$	1	$-\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_{31}$	$\mathbf{e}_{12}$	1	$\mathbf{e}_1$
$\boldsymbol{\Omega}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_1$	1

(b) The Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_3$.

$a \circledast b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$
$\mathbf{e}_1$	$\mathbf{e}_1$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_2$	$\mathbf{e}_3$	$\boldsymbol{I}$	$-\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	-1	$\mathbf{e}_{23}$	$\mathbf{e}_1$	$\boldsymbol{I}$	$-\mathbf{e}_3$	$-\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	-1	$\boldsymbol{I}$	$-\mathbf{e}_1$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\boldsymbol{I}$	-1	$\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$\mathbf{e}_3$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$-\mathbf{e}_3$	$-\boldsymbol{I}$	$\mathbf{e}_1$	$-\mathbf{e}_{23}$	-1	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\boldsymbol{I}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	-1	$\mathbf{e}_1$
$\boldsymbol{I}$	$\boldsymbol{I}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	-1

(c) The Multiplication Table for the Modified Octonion Algebra $\mathbb{O}_m$ isomorphic to the Octonion Algebra $\mathbb{O}$.

$a \circledast b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\overline{\mathbf{e}_{12}}$	$\overline{\mathbf{e}_{31}}$	$\overline{\mathbf{e}_{23}}$	$\boldsymbol{I}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\overline{\mathbf{e}_{12}}$	$\overline{\mathbf{e}_{31}}$	$\overline{\mathbf{e}_{23}}$	$\boldsymbol{I}$
$\mathbf{e}_1$	$\mathbf{e}_1$	-1	$-\overline{\mathbf{e}_{12}}$	$\overline{\mathbf{e}_{31}}$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$-\boldsymbol{I}$	$\overline{\mathbf{e}_{23}}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$\overline{\mathbf{e}_{12}}$	-1	$-\overline{\mathbf{e}_{23}}$	$-\mathbf{e}_1$	$-\boldsymbol{I}$	$\mathbf{e}_3$	$\overline{\mathbf{e}_{31}}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$-\overline{\mathbf{e}_{31}}$	$\overline{\mathbf{e}_{23}}$	-1	$-\boldsymbol{I}$	$\mathbf{e}_1$	$-\mathbf{e}_2$	$\overline{\mathbf{e}_{12}}$
$\overline{\mathbf{e}_{12}}$	$\overline{\mathbf{e}_{12}}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\boldsymbol{I}$	-1	$-\overline{\mathbf{e}_{23}}$	$\overline{\mathbf{e}_{31}}$	$-\mathbf{e}_3$
$\overline{\mathbf{e}_{31}}$	$\overline{\mathbf{e}_{31}}$	$\mathbf{e}_3$	$\boldsymbol{I}$	$-\mathbf{e}_1$	$\overline{\mathbf{e}_{23}}$	-1	$-\overline{\mathbf{e}_{12}}$	$-\mathbf{e}_2$
$\overline{\mathbf{e}_{23}}$	$\overline{\mathbf{e}_{23}}$	$\boldsymbol{I}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$-\overline{\mathbf{e}_{31}}$	$\overline{\mathbf{e}_{12}}$	-1	$-\mathbf{e}_1$
$\boldsymbol{I}$	$\boldsymbol{I}$	$-\overline{\mathbf{e}_{23}}$	$-\overline{\mathbf{e}_{31}}$	$-\overline{\mathbf{e}_{12}}$	$\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_1$	-1

(d) The Multiplication Table for the Octonion Algebra $\mathbb{O}$.

Table 6.2: The Multiplication Tables for the Four Algebras.

$a \circ b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$\boldsymbol{I}$	$\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_1$	$\boldsymbol{I}$	$\mathbf{e}_3$	$\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\boldsymbol{I}$	$\mathbf{e}_1$	$-\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\boldsymbol{I}$	-1	$\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_3$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$\boldsymbol{I}$	$-\mathbf{e}_1$	$-\mathbf{e}_{23}$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_2$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	-1	$-\mathbf{e}_1$
$\boldsymbol{I}$	$\boldsymbol{I}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	-1

(a) The Multiplication Table for the Clifford Algebra $\mathcal{C}\ell_3$.

$a \circledcirc b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
$\mathbf{e}_1$	$\mathbf{e}_1$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$\boldsymbol{\Omega}$	$-\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	-1	$\mathbf{e}_{23}$	$-\mathbf{e}_1$	$\boldsymbol{\Omega}$	$\mathbf{e}_3$	$-\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	-1	$\boldsymbol{\Omega}$	$\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\boldsymbol{\Omega}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$\mathbf{e}_3$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$\boldsymbol{\Omega}$	$-\mathbf{e}_1$	$-\mathbf{e}_{23}$	1	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_1$
$\boldsymbol{\Omega}$	$\boldsymbol{\Omega}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_1$	-1

(b) The Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_{0,3}$.

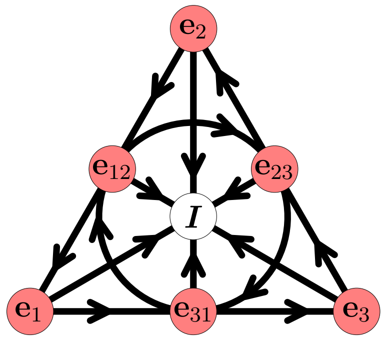
$a \circledast b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{I}$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$\boldsymbol{I}$	$\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_1$	$\boldsymbol{I}$	$\mathbf{e}_3$	$\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\boldsymbol{I}$	$\mathbf{e}_1$	$-\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\boldsymbol{I}$	-1	$-\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$-\boldsymbol{I}$	$-\mathbf{e}_1$	$\mathbf{e}_{23}$	-1	$-\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\boldsymbol{I}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_{31}$	$\mathbf{e}_{12}$	-1	$\mathbf{e}_1$
$\boldsymbol{I}$	$\boldsymbol{I}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	1

(c) The Multiplication Table for the Modified Antioctonion Algebra $\overline{\mathbb{O}}_m$ Isomorphic to the Antioctonion Algebra $\overline{\mathbb{O}}$

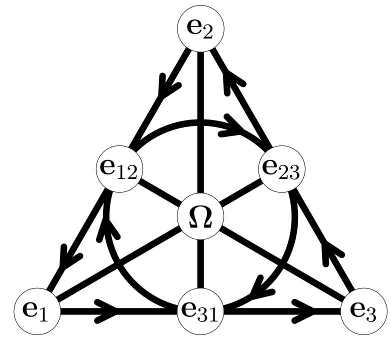
$a \circledast b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\overline{\mathbf{e}_{12}}$	$\overline{\mathbf{e}_{31}}$	$\overline{\mathbf{e}_{23}}$	$\boldsymbol{I}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\overline{\mathbf{e}_{12}}$	$\overline{\mathbf{e}_{31}}$	$\overline{\mathbf{e}_{23}}$	$\boldsymbol{I}$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$-\overline{\mathbf{e}_{12}}$	$\overline{\mathbf{e}_{31}}$	$-\mathbf{e}_2$	$\mathbf{e}_3$	$-\boldsymbol{I}$	$-\overline{\mathbf{e}_{23}}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$\overline{\mathbf{e}_{12}}$	1	$-\overline{\mathbf{e}_{23}}$	$\mathbf{e}_1$	$-\boldsymbol{I}$	$-\mathbf{e}_3$	$-\overline{\mathbf{e}_{31}}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$-\overline{\mathbf{e}_{31}}$	$\overline{\mathbf{e}_{23}}$	1	$-\boldsymbol{I}$	$-\mathbf{e}_1$	$\mathbf{e}_2$	$-\overline{\mathbf{e}_{12}}$
$\overline{\mathbf{e}_{12}}$	$\overline{\mathbf{e}_{12}}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$\boldsymbol{I}$	-1	$\overline{\mathbf{e}_{23}}$	$-\overline{\mathbf{e}_{31}}$	$-\mathbf{e}_3$
$\overline{\mathbf{e}_{31}}$	$\overline{\mathbf{e}_{31}}$	$-\mathbf{e}_3$	$\boldsymbol{I}$	$\mathbf{e}_1$	$-\overline{\mathbf{e}_{23}}$	-1	$\overline{\mathbf{e}_{12}}$	$-\mathbf{e}_2$
$\overline{\mathbf{e}_{23}}$	$\overline{\mathbf{e}_{23}}$	$\boldsymbol{I}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\overline{\mathbf{e}_{31}}$	$-\overline{\mathbf{e}_{12}}$	-1	$-\mathbf{e}_1$
$\boldsymbol{I}$	$\boldsymbol{I}$	$\overline{\mathbf{e}_{23}}$	$\overline{\mathbf{e}_{31}}$	$\overline{\mathbf{e}_{12}}$	$\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_1$	1

(d) The Multiplication Table for the Antioctonion Algebra $\overline{\mathbb{O}}$.

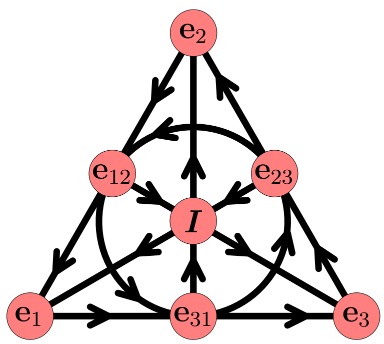
Table 6.3: The Multiplication Tables for the Four Antialgebras.



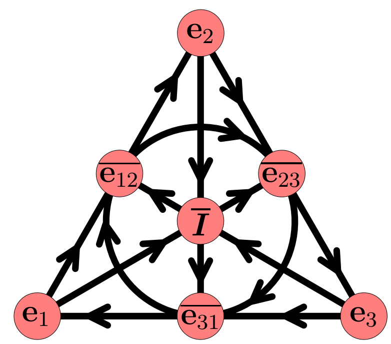
(a) The RPSG for the Clifford Algebra $\mathcal{C}\ell_{0,3}$.



(b) The RPSG for the Orientation Congruent Algebra $\mathcal{OC}_{3,0}$.

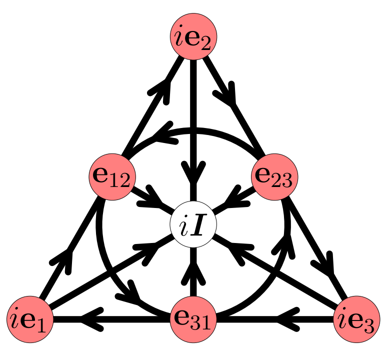


(c) The RPSG for the Modified Octonion Algebra $\mathbb{O}_m$.

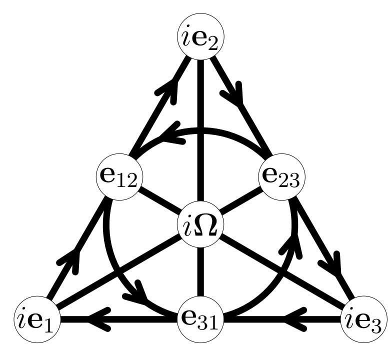


(d) The RPSG for the Octonion Algebra $\mathbb{O}$.

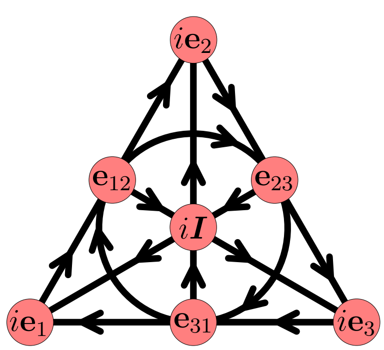
Figure 6.4: The Reduced Product Sequence Graphs for Four Algebras that are Directly Representable as PSGs.



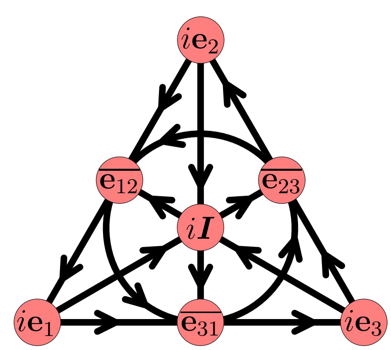
(a) The RPSG for the Algebra $\mathcal{C}\ell'_{3,0} \cong (\mathcal{C}\ell_{3,0})^{\mathrm{op}}$.



(b) The RPSG for the Algebra $\mathcal{OC}'_{0,3} \cong (\mathcal{OC}_{0,3})^{\mathrm{op}}$.



(c) The RPSG for the Algebra $\overline{\mathbb{O}}'_m \cong (\overline{\mathbb{O}}_m)^{\mathrm{op}}$.



(d) The RPSG for the Algebra $\overline{\mathbb{O}}' \cong \overline{\mathbb{O}}^{\mathrm{op}}$.

Figure 6.5: The Reduced Product Sequence Graphs for the Rectified Forms of the Antialgebras of the Four Above.

7 Multiplication Tables: Symmetries, Matrices, and Functions

In this section we continue the comparison of the Clifford and the orientation congruent algebra multiplication tables that we began in subsection 2.5. We define several canonical forms for these tables and examine their resulting symmetries. The canonical forms for the Clifford multiplication tables, based on the Gray code order of the elements in the indicial row and column, have already been reported by other investigators. Those for the orientation congruent tables appear to be new. Defining the $\mathcal{OC}$ multiplication table canonical forms requires rather involved and interesting combinatorial expressions. The $\mathcal{OC}$ table sign distributions create some striking patterns (see Fig. 7.1).

As is already known the sign patterns in the Clifford algebra multiplication tables generate various forms of Hadamard matrices and these matrices may be interpreted as particular ordered sequences of Walsh functions. But the sign patterns in the orientation congruent algebra tables generate some different kinds of matrices that may also be interpreted as sequences of orthogonal functions. This connection seems to be currently unknown.

$\boxed{\text{TO BE DEVELOPED FURTHER}}$

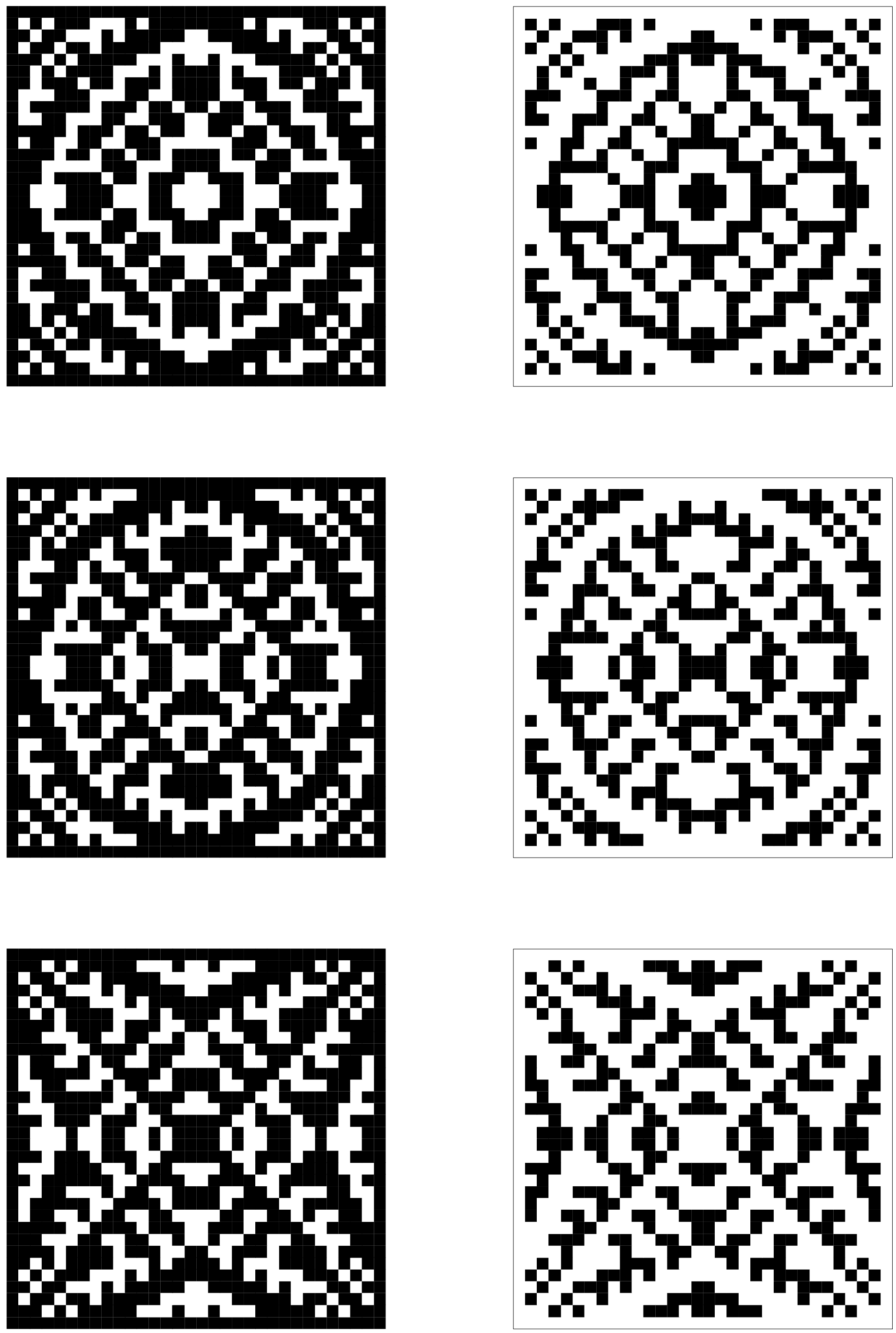


Figure 7.1: **Some Pretty Multiplication Table Sign Patters for $\mathcal{OC}_5$.**

8 Specific Associativity and Associomediativity

8.1 The Null Associator Predictor

The *associator* of any three elements of an algebra $[u, v, w] = (uv)w - u(vw)$ is null exactly when u, v, w is a *specifically associative* triple. The γ in the next theorem is the *grade of* operator. The symbol $\cap$ represents the *meet* operator.

Theorem 8.1 *In a $\mathcal{OC}$ algebra a triple of blades A, B, C has null a associator if and only if the following* null associator predictor *is even:*

$$[\gamma(A\cap C) + \gamma(A)\,\gamma(C)]\,[\gamma(A\cap B) + \gamma(B\cap C)] \\ + \gamma(A\cap C)\,\gamma(B)[\gamma(A) + \gamma(C)]. \quad (8.1)$$

Proof. A lot of tedious algebraic manipulation of sign factor functions that is left to the reader or the reader's symbolic computer algebra system. ■

Note the symmetry in this expression. The A and C at the extreme positions of the associator always appear in balanced pairs that commute.

8.2 The Associomediative Property of Counits

The counits in an orientation congruent algebra appear to have (under certain conditions given below) what we will call the *associomediative property*: If a sequence of counits are interleaved into an expression of products of blades that expression becomes *freely associative*—parentheses may be added in any well-formed way without changing the value of the expression.

This is a very interesting statement. But at the moment it remains a *conjecture* having been verified only empirically using Mathematica for $\mathcal{OC}_n$ with $n \leq 11$. In these computational tests the Euclidean orientation congruent algebra for each order n was challenged with all combinations of grades of two blades and with all possible grades of products between them. The author has not attempted a proof. She can give no estimate of its difficulty (assuming the conjecture is correct).

The Associomediative Conjecture

Let $A, B \in \mathcal{OC}_{p,q}$ be blades and let $\boldsymbol{\omega}_{\mathscr{A}}$ be a counit of $\mathscr{A}$ where $\mathscr{A} = \{A, B\} \cup \mathscr{C}$ for any $\mathscr{C} \subseteq \mathcal{OC}_{p,q}$. Let any unparenthesized expression which is a substring of finite length $r \geq 1$, beginning with "A" but not ending with "$\circledcirc$", taken from any string formed by concatenating a finite number of copies of the string "$A \circledcirc \boldsymbol{\omega}_{\mathscr{A}} \circledcirc B \circledcirc \boldsymbol{\omega}_{\mathscr{A}} \circledcirc$" be called an "associomediative expression in A, B, and $\boldsymbol{\omega}_{\mathscr{A}}$ of order s" *where $s = \frac{1}{2}(r+1)$ is the number of factors in the $\mathcal{OC}$ multiproduct that is the expression.*

Also let γ be the grade of *function just defined in the subsection above. Then for all $\mathcal{OC}_{p,q}$ and all blades $A, B \in \mathcal{OC}_{p,q}$ and all counits $\boldsymbol{\omega}_{\mathscr{A}} \in \mathcal{OC}_{p,q}$ satisfying the definition of an associomediative expression, all parenthesizations* *into binary products of the associomediative expression in the same A, B, and $\boldsymbol{\omega}_{\mathscr{A}}$ of the same order s are equal if and only if either $s \leq 8$ or*

$$\gamma(A)\,\gamma(B) + \frac{1}{2}[\gamma(A) + \gamma(B) - \gamma(A \circledcirc B)] \tag{8.2}$$

is even.

$\boxed{\text{TO BE DEVELOPED FURTHER}}$

9 Matrix Representations of the Orientation Congruent Algebra

In subsection 6.1 we gave the multiplication table (Tab. 6.1) of a small nonassociative algebra $\mathcal{U}$ with basis elements 1, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. This algebra $\mathcal{U}$ is isomorphic to the (imperfect) orientation congruent algebra $\mathcal{OC}_2$ with the correspondences $1 \leftrightarrow 1$, $\mathbf{a} \leftrightarrow \mathbf{e}_1$, $\mathbf{b} \leftrightarrow \mathbf{e}_2$, $\mathbf{c} \leftrightarrow \mathbf{e}_{12}$.

A kind of matrix representation of $\mathcal{OC}_2$ may also be established. We claim the matrices

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{9.1}$$

are representations of the basis blades of $\mathcal{OC}_2$ under the correspondences $\mathbf{I} \leftrightarrow 1$, $\mathbf{A} \leftrightarrow \mathbf{e}_1$, $\mathbf{B} \leftrightarrow \mathbf{e}_2$, $\mathbf{C} \leftrightarrow \mathbf{e}_{12}$.

But under what product? Let the set of square matrices and their matrix algebras over $\mathbb{R}$, $\mathbb{C}$, and Hamilton's quaternions $\mathbb{H}$, be written as $\mathrm{Mat}(\mathbb{R}, n)$, $\mathrm{Mat}(\mathbb{C}, n)$, and $\mathrm{Mat}(\mathbb{H}, n)$, respectively. As is well known, it is among these algebras that the standard faithful matrix representations of the Clifford algebras $\mathcal{C}\ell_{p,q}$ are found. All elements of these matrix algebras do associate because their multiplication is the usual matrix product, but the elements of $\mathcal{OC}_2$ do not associate under the orientation congruent product. Therefore the usual matrix algebras and their product cannot represent the orientation congruent product of $\mathcal{OC}_2$.

Instead we define the (left) *Hermitian conjugate product,* denoted by a circled star $\circledast$, so that for all conforming matrices $\mathbf{P}$ and $\mathbf{Q}$

$$\mathbf{P} \circledast \mathbf{Q} \equiv \mathbf{P}^{\mathrm{H}}\mathbf{Q} \equiv \overline{\mathbf{P}^{\mathrm{t}}}\mathbf{Q}. \tag{9.2}$$

Here juxtaposition indicates the standard associative matrix product, the overbar indicates matrix complex conjugation, the superscript lower case t indicates matrix transposition, and the superscript upper case H indicates Hermitian conjugation. The reader may verify that under this nonassociative matrix product the algebra of the matrices $\mathbf{I}$, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ taken from $\mathrm{Mat}(\mathbb{C}, 2)$ is isomorphic to $\mathcal{OC}_2$.

The author has found nonassociative matrix algebra representations for $\mathcal{OC}_3$ and some other orientation congruent algebras by ad hoc methods. It appears that if the faithful matrix representation of the Clifford algebra $\mathcal{C}\ell_{p,q}$ requires matrices taken from $\mathrm{Mat}(\mathbb{R}, n)$, $\mathrm{Mat}(\mathbb{C}, n)$, or $\mathrm{Mat}(\mathbb{H}, n)$, the smallest nonassociative matrix representation of $\mathcal{OC}_{p,q}$ requires matrices taken from $\mathrm{Mat}(\mathbb{R}, 2n)$, $\mathrm{Mat}(\mathbb{C}, 2n)$, or $\mathrm{Mat}(\mathbb{H}, 2n)$, respectively.

TO BE DEVELOPED FURTHER

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