Unspecified Journal
Volume 00, Number 0, Pages 000–000
S ????-????(XX)0000-0

EXTERIOR CALCULUS IN THE IMAGE OF ODD FORMS WITH THE ORIENTATION CONGRUENT ALGEBRA

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For Elaine Yaw in honor of friendship

Abstract. *Odd* (*twisted*) differential forms are naturally endowed with two transversely-oriented parts: a generalized sign and magnitude. On oriented manifolds, odd forms may be reduced to *even* ones. Also, W.L. Burke has modeled odd forms with an unnatural two-part structure. Neither approach suffices if odd forms are pulled back between manifolds with an odd difference in dimensions. Then the new *native exterior calculus* and *orientation congruent (OC) algebra* (a Clifford-like, noncommutative Jordan algebra) must be used to resolve Burke's dilemma: altering the natural orientation rule for either pullback or integration. I review the work of K. Warnick et al. on electromagnetic boundary conditions and G. Marmo et al. on the apparently inconsistent parities of electromagnetic quantities due to space-time vs. space orientations.

Received by the editors September 18, 2008.

2000 *Mathematics Subject Classification.* Primary 58A10; Secondary 15A66, 15A75, 15A78, 17A15, 17D99, 20N05, 35F15, 35Q60, 51A05, 51N15.

For some relief from my duties at the East Lansing Food Coop, I thank my coworkers Lindsay Demaray, Liz Kersjes, and Connie Perkins, nee Summers. For encouragement and financial assistance, I thank DeeAnna Wooden and Elaine Yaw. For providing a copy of Marmo et al. [128] I thank, Daniel Henry Gottlieb. For professional kind words, interest, and encouragement, I thank John Browne and John Masterson. Finally, for demonstrating the value of hard work, I thank an anonymous manager.

Contents

1. Introduction

> The importance of twisted tensors in physics has been neglected by nearly everyone.
>
> William L. Burke [34, p. xiii]

Twisted tensors, particularly as differential forms of odd type, usually *can* be ignored in most applications. In fact, using the exterior calculus equivalent of the right hand rule, they can be converted into ordinary, even forms. However, physical insight is lost in so abandoning the essential qualities of odd forms. On the other hand, the theoretical framework required to fully and naturally represent odd forms is more complicated.

For those workers choosing to use odd forms, William L. Burke's approach is an attractive compromise. In fact, Burke's publications [33, 34, 36] are the starting point for this work. Sadly I relate that in 1996 the astrophysicist William Lionel Burke died prematurely at age 55 [199]. Had that not been so, perhaps he would have authored a paper such as this long ago.

Burke incorporated odd forms in exterior calculus by representing them as an ordered pair $(d\alpha, \Omega)$, where $d\alpha$ is an ordinary even form and Ω is a top-dimensional n-form. The extended exterior calculus based on Burke's representation of odd forms is sufficient for most applications. But *not* when such forms are pulled back by mappings between manifolds whose dimensions differ by an odd number.

To handle such mappings between manifolds with an odd difference in their dimensions, I develop the new *native exterior calculus* by symbolizing *both* odd and even differential forms in what I call the *native representation*. The native representation and its exterior calculus are based on and extend Burke's work. However, the new theory fully respects the essential, internally-complementary, directionally-bipartite, sign and magnitude nature of odd forms as it operates on them.

That odd geometric quantities have a natural two-part complementary, sign and magnitude directional aspect is not a new realization. In fact, illustrations of odd multiforms or multivectors that are the graphical counterpart of the native representation have been published as early 1924 in Schouten's German language book [158, p. 22] and have continued to appear in the literature ever since.[1] What is remarkable is that it has taken until now for the corresponding symbolic representation and its theoretical underpinnings to be worked out.

For an elementary example of the power of the native exterior calculus, I treat the discontinuous electromagnetic boundary conditions (the so-called *jump conditions*) produced by a surface that is charged or carrying a current. About ten years ago this situation puzzled both Burke [36], [195, p. 332, fn.], and another group of authors Warnick, Selfridge, and Arnold [195]. A researcher who uses Burke's exterior calculus to analyze the jump conditions is forced to make one of two ad hod modifications: alter the natural orientation rule either for pullback or for integration. Burke chose the first modification, while Warnick et al. chose the second. However, neither is necessary with the native exterior calculus.

[1] Here are some citations that contain pictures of odd forms in their native representation: the paper of Schouten and van Dantzig [164]; Schouten's books [161, p. 28] and [162, pp. 31–33, 55]; Bossavit's on-line book [23, p. 72]; Jancewicz's papers [105, 106, 108]; the book by Hehl and Obukhov [89, p. 145]; and the papers coauthored with Hehl [88, 90].

The same shortcomings of Burke's extension of exterior calculus to odd forms also seem to afflict the other methods of representing and manipulating odd differential forms and multivectors. I say this because I know of no other theory that completely captures the inherent two-part, sign and magnitude, self-complementary direction of odd quantities. A hint of how widespread the suspected general situation might be is found in the next example from special relativity.

Although the electromagnetic dilemma reported by Warnick et al. involves a mapping from a manifold of dimension 3 (ordinary space that contains electromagnetic fields) to a manifold of dimension 2 (a surface that carries an electrical charge or current), its cause—a map with an odd dimensional change—is the same as that of the conundrum that Marmo et al. [128, 129] attempted to resolve with the exterior calculus extended in a common, but not very revealing, way to odd differential forms. These authors analyzed the apparent shift in parities of electromagnetic fields that occurs when 4-dimensional spacetime is split by an observer into one time and three space dimensions. I cannot do it here, but hope to apply the native exterior calculus to this problem in a future publication.

Burke's $(d\alpha, \Omega)$ ordered pair expression for odd forms is cognate to what is called in this paper the *extremum representation* for both odd and even forms. In a draft paper [36] which may have been his last publication, Burke described a new representation for odd forms. An intermediate expression in his description, which, unfortunately, Burke never exploited, inspired another representation of odd and even forms presented in this paper, the *correlated representation*. Interrelating the exterior products of odd and even forms in the extremum and correlated representations leads to the new *orientation congruent algebra* (or $\mathcal{OC}$ algebra). The orientation congruent algebra is required to form the exterior algebras and calculi of these two representations and also the native one.

By abstracting from our first explorations of the orientation congruent algebra, we formulate an axiom system for it that is similar to the axiom system for Clifford algebra based on *generators and relations*. This type of axiom system is common in the Clifford algebra literature under the name for Clifford algebra interpreted geometrically, *geometric algebra*.

The orientation congruent algebra is also a *Clifford-like* algebra. Clifford-likeness may be defined by considering products of the basis multivectors that are derived from an orthonormal set of basis vectors. Then the Clifford-likeness of some algebra means that the product of two given basis multivectors in that algebra is same as their Clifford product *up to sign*.

An explicit Clifford-like formula for the $\mathcal{OC}$ product is provided later. This formula determines the *sign factor*, $\sigma = \pm 1$, that, when applied to the Clifford product, converts it to the orientation congruent product. The sign factor may be expressed as a function of the degrees (or *grades* in Clifford algebra jargon) of the multiplier and multiplicand of a given orientation congruent product.

In practical application, humans or computer programs that can calculate the Clifford product can also calculate the orientation congruent product by using the sign factor. More importantly, in theoretical application, by reducing the abstract algebraic theorems of the orientation congruent algebra to the ordinary algebraic manipulations used to calculate the sign factor combined with the known theorems of Clifford algebra, the Clifford-like formula for the $\mathcal{OC}$ product becomes the vehicle for the development of most of this work.

From the perspective of abstract algebra, the $\mathcal{OC}$ algebra is one of a large class of nonassociative algebras known as *noncommutative Jordan* algebras. In addition, the Cayley or multiplication table of products of basis multivectors (as defined in the previous paragraph) and their negatives defines a *quasigroup with identity* (also known as a *loop*) which appears to have unique properties. However, a thorough investigation of the algebraic properties of the orientation congruent algebra and its derived loop will have to wait until another publication.

Amazingly, this work is connected to special relativity in yet another way than the one mentioned above as analyzed by Marmo et al. [128, 129]. Namely, in the orientation congruent algebra for a Euclidean 3-dimensional space, denoted $\mathcal{OC}_3$, the three elements formed from the orientation congruent products of two orthonormal basis vectors are isomorphic to the so called *hyperbolic quaternions* of Alexander MacFarlane (see [200]). This connection is also related to Abraham A. Ungar's approach to special relativity and hyperbolic geometry with *gyrogroups* expounded in his books [180, 181] (and many papers, not cited here). However, these connections cannot be pursued further at this time.

Intended Audience. In this paper I touch on topics from diverse mathematical specialties, abstract and applied: differential geometry, oriented and unoriented projective geometry, groups and loops, associative and nonassociative algebras, Clifford and geometric algebra, and electromagnetic theory. Some results from these diverse areas may be well known to specialists in any one of these fields, but all of them are unlikely to be known to any single reader, especially a reader in my intended audience—those physicists and engineers using applied differential geometry, but who, typically, are not familiar with odd differential forms. Therefore, specialists be warned: background material is frequently discussed in more detail than would be palatable to you.

General Prerequisites. The general background for this paper is found in the following works: for applied differential geometry, the publications of William Burke [33, 34, 36]; for Clifford and geometric algebra, the book by Pertti Lounesto [123], the tome of David Hestenes and Garret Sobczyk [97], and the excellent synopsis of Richard E. Harke [84]. Drawings of all the three-dimensional odd and even quantities (multiforms or multivectors) and examples of their use to represent specific physical quantities are found in Bernard Jancewicz's papers [105, 106].

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William Burke's publications [33, 34, 36] are the starting point for this work. First, in a 1983 paper [33], then, in a 1985 textbook [34], he presented an excellent first pass at adapting Cartan's exterior calculus to *odd* (or *twisted*) differential forms. His technique is based on the above mentioned $(d\alpha, \Omega)$ ordered pair expression for odd forms, where $d\alpha$ is an ordinary even form and Ω is a top-dimensional n-form. This expression and Burke's calculation rules for the resulting modified exterior calculus are essentially equivalent to this paper's *extremum representation* and *extremum exterior calculus* of both odd and even differential forms.

Later, on the first page of a draft paper [36], Burke describes his new, self-named *William's Twisted Notation* for odd forms as "what I have found to be the best and simplest notation for twisted forms." Then, on pages 5–6, he explains how to transform the $(d\alpha, \Omega)$ representation into his Twisted Notation. It is the penultimate expression in this transformation (unfortunately, not exploited by Burke) that opens the door to all the results presented here. This expression directly inspired this paper's *correlated representation* and its *correlated exterior calculus* of both odd and even differential forms.

Experience shows that Burke's techniques based on expressing odd forms as the ordered pair $(d\alpha, \Omega)$ are almost always sufficient. However, they fail when odd differential forms are mapped (pulled back) between manifolds whose dimensions differ by an odd integer. This situation requires, not the Burkean adaptation of Cartan's calculus, but this paper's new *native representation* and *native exterior calculus* of both odd and even differential forms.[2]

An elementary example involving such a map is the formulation of the jump conditions of the odd electric flux density 2-form D and the odd magnetic field intensity 1-form H due to surface sources. Burke treated it repeatedly: first, in his 1983 paper [33]; then, in his 1985 textbook [34]; and finally, in his 1995 draft paper [36], one of his last works before his premature death.[3]

Burke's methods have also been used by Warnick, Selfridge and Arnold in their 1995 paper [195] on the electromagnetic jump conditions. This paper introduced their novel *boundary projection operator*.[4] However, the contribution of Warnick et al. that is relevant here is their clear, acute dissection of the inadequacies of Burke's approach to the jump conditions.

In this electromagnetic example, the Cartan-Burke analyst, attempting to maintain the commutation of exterior differentiation with pullback, faces the dilemma of altering the natural orientation rule for either integration or for pullback. Although it would be applied to only the pullback from 3-space to a surface, the first choice initiates an undesirable policy of ad hoc change. On the other hand, the second choice awkwardly requires that forms of different degrees be treated differently—violating the spirit of Cartan. Yet this dilemma is unnecessary once the Cartan-Burke exterior derivative is generalized to the native exterior derivative.

Using the native exterior calculus we naturally resolve the above dilemma that confronted first Burke and later Warnick, Selfridge, and Arnold (and also again Burke whose private communication they cite [195, p. 332, fn.]) in their attempt to formulate these electromagnetic field jump conditions. In our analysis at the end of this paper, unlike Burke, we do not modify the natural orientation rule for pullback to a surface, and unlike Warnick, Selfridge, and Arnold, we do not modify the natural orientation rule for integration.

Although the electromagnetic dilemma reported by Warnick et al. involves a mapping from a manifold of dimension 3 (ordinary space that contains electromagnetic fields) to a manifold of dimension 2 (a surface that carries an electrical charge or current), its cause—a map with an odd dimensional change—is the same as that of the conundrum that Marmo et al. [128, 129] attempted to resolve with the exterior calculus extended in a common, but not very revealing, way to odd differential forms. These authors analyzed the apparent shift in parities of electromagnetic

[2]The same shortcoming would seem to afflict the other methods of representing and manipulating odd differential forms. Although I have not verified it, I say this because I know of no other theory that completely captures the essential internally-complementary, directionally-bipartite, sign and magnitude nature of odd quantities.

[3]Unfortunately, William Lionel Burke died at age 55 in 1996 from a cervical fracture that he suffered in an automobile accident. See his Wikipedia entry [199] for more information.

[4]Since the fall of 1995, these pioneering authors have been teaching undergraduate electrical engineering students using a curriculum based on differential forms [196, p. 54].

fields that occurs when 4-dimensional spacetime is split by an observer into one time and three space dimensions. I cannot do it here, but hope to apply the native exterior calculus to this problem in a future publication.

Later in this paper we develop the three above mentioned types of symbolic representations. They may be used not only for odd forms, but also for even forms, and odd and even multivectors. These representations are actually equivalence classes of ordered pairs with each part of the pair expressed, in general, as a sum of forms or multivectors. Recapitulating, they are the *extremum* (short for *unbound extremum*), the *correlated* (short for *bound correlated*), and the *native* (or *unbound correlated*) representations.

We may briefly characterize these three representations as follows. As mentioned above, the extremum representation is equivalent to Burke's ordered pair representation $(d\alpha, \Omega)$ where $d\alpha$ is an ordinary even differential form and Ω is a top-dimensional n-form or *volume form*. It also appears to be essentially equivalent to all previously known representations of odd quantities. The correlated representation is an intermediate form that is helpful to the development. It is nearly equivalent to the native representation. The native representation is the natural representation for odd forms, but it also accommodates even forms. It totally captures the inherent, self-complementary, two-part, sign and magnitude direction of odd forms. Both the extremum and correlated representations are of limited validity, but the native representation is generally valid.

Later in this work, we meet the *orientation congruent* ($\mathcal{OC}$) *algebra*. It necessarily appears when we construct the exterior algebras of the extremum and correlated representations, and relate the two. However, it is essential throughout the theory. By abstracting from our first explorations of the $\mathcal{OC}$ algebra, we formulate an axiom system similar to the one for Clifford algebra based on *generators and relations* that is common in the literature.

The orientation congruent algebra is also a *Clifford-like* algebra. Clifford-likeness may be defined by considering products of the basis multivectors that are derived from an orthonormal set of basis vectors. Then the Clifford-likeness of some algebra means that the product of two given basis multivectors in that algebra is same as their Clifford product *up to sign*.

An explicit Clifford-like formula for the $\mathcal{OC}$ product is provided later. This formula determines the *sign factor* (± 1) that, when applied to the Clifford product, converts it to the orientation congruent product. The sign factor is a function of the degrees (or *grades* in Clifford algebra jargon) of the multiplier and multiplicand of a given product.

In practical application, humans or computer programs that can calculate the Clifford product can also calculate the orientation congruent product by using the sign factor. More importantly, in theoretical application, by reducing the abstract algebraic theorems of the orientation congruent algebra to the ordinary algebraic manipulations used to calculate the sign factor combined with the known theorems of Clifford algebra, the Clifford-like formula for the $\mathcal{OC}$ product becomes the vehicle for the development of most of this work.

From the perspective of abstract algebra, the $\mathcal{OC}$ algebra is one of a large class of nonassociative algebras known as *noncommutative Jordan* algebras. In addition, the Cayley or multiplication table of products of basis multivectors (as defined in the previous paragraph) and their negatives defines a *quasigroup with identity* (also known as a *loop*) which appears to have unique properties. However, a thorough investigation of the algebraic properties of the orientation congruent algebra and its derived loop will have to wait until another publication.

Amazingly, this work is connected to special relativity in yet another way than the one mentioned above as analyzed by Marmo et al. [128, 129]. Namely, in the orientation congruent algebra for a Euclidean 2-dimensional space, denoted $\mathcal{OC}_2$, any orthonormal basis and its products is a set of elements isomorphic to the so called *hyperbolic quaternions* of Alexander MacFarlane (see [200]). This connection is related to Abraham A. Ungar's approach to special relativity and hyperbolic geometry with *gyrogroups* expounded in his books [180, 181] (and many papers, not cited here). However, these connections cannot be pursued further at this time.

Some of the symbolic representations developed in this paper have appeared in various publications of other authors as cognates that are limited to odd forms and multivectors. However, I do not know of any earlier systematic development of all three representations for both odd and even quantities. Furthermore, nowhere else have I found the correlated and native representations exploited as the basis for an exterior algebra or calculus as is done here. Even so, it is remarkable that illustrations of odd quantities that are the graphical counterparts of the native representation have been published as early 1924 in Schouten's German language book [158, p. 22] and have continued to appear in the literature ever since.

General Prerequisites. Quite enough new material is introduced in this paper. Therefore, to keep it to a practical length, I assume the reader has a certain background. More or less, you should be familiar with odd differential forms as they appear in the publications of William Burke [33, 34, 36], and with Clifford algebra (or geometric algebra) as it is treated in the book by Pertti Lounesto [123] and the excellent synopsis of Richard E. Harke [84]. More physical motivation for odd quantities (multiforms or multivectors) can be found in Bernard Jancewicz's papers [105, 106].

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Odd (or twisted) differential forms (and multivectors) are rather easy to understand and visualize. Although, in the following discussion it is best to consider only the simple ones, that is, those that may be represented by a product of one-forms. Then, unlike even (or ordinary) differential forms (and multivectors) odd differential forms naturally have two parts both of which have directional properties. One of the "directions" of such an odd form may be associated with an ordinary form if we keep the magnitude or measuring property of the associated form, that is, its ability to return a number when acting on a multivector of the same degree, but ignore the positive or negative sign of that number. Graphically, it may be represented by a differential form stripped of the arrows that indicate the sense of its direction.

It is the other direction that determines the sign when an odd form contracts with an odd multivector. It may also be represented by a differential form, but now one whose magnitude or measuring property is ignored—only the sign of the result of its contraction with a multivector is relevant. Graphically, it may be represented by a multivector that is transverse to the other magnitude-determining part of the odd form, that is, one whose contraction with other part of the odd form is zero.

Odd (or twisted) differential forms remain unknown to many mathematicians, physicists, and engineers—even some familiar with differential geometry. However, any researcher deeply interested in electrodynamics eventually discovers these two truths: Odd differential forms arise naturally alongside the ordinary even ones. Both are necessary to fully model the field quantities of this and other physical theories.

If odd forms are so innate and essential in physics why does this ignorance continue? The status quo remains since a full and direct understanding of the property that describes whether a differential form is odd or even—its *orientation*—can either be sometimes ignored or be altogether avoided with makeshift techniques. Thus, absent or garbled presentations of odd quantities sidestep the more complicated theoretical machinery necessary for their full and direct analysis. In spite of this criticism, any approach to orientation taken within some discipline is, of course, broadly shaped by the associated forces and has merit within that context.

Various treatments are found in traditional physics and engineering, pure mathematics, and applied mathematics. Let us quickly review some common approaches from these fields ending with the one which is the root of this paper.

The traditional physics and engineering treatment of odd quantities uses the Gibbs-Heaviside vector calculus or indexed tensor notation. In vector calculus the orientation of an odd vector, such as that representing the magnetic field intensity, is disguised by converting it to an axial vector, which in this example would be $\mathbf{H}$. Worse, because the vector cross product is defined by assigning a reference orientation to ordinary three-dimensional space, the Gibbs-Heaviside vector calculus also converts even bivectors, such as that representing the magnetic flux density, to an axial vector, which in this example would be $\mathbf{B}$. In tensor notation the oddness of a quantity is almost completely obscured by index notation and manifests itself essentially in the transformational properties of the tensorial representation.

Physicists and engineers attempt to compensate for the deficiencies of vector calculus by adopting labels such as *axial vector* and *pseudovector*. They attempt to cope with those of tensor notation with a special set of rules for transforming odd quantities together with labels such as *pseudotensor*, *bivector density*, or *W-vector*, or with modifications of the kernel (the symbol to which the indices are attached).

In pure mathematics, orientation usually only comes up in topological discussions as a certain property of a manifold: its orientability or nonorientability. Twisting by the line bundle. This is a powerful approach, however it is not as intuitive as the one provided by some authors writing in applied differential geometry. That is the starting point for this paper.

In applied mathematics, some more modern physics and engineering practitioners of differential geometry come closer to fully and directly treating the orientation of an odd quantities by using a symbolic representation that, at least, makes their *bipartite directional nature* manifest. For odd differential forms this is the ordered pair representation (α, Ω'). Here α is an ordinary even form, while $\Omega' = \pm\Omega$ is a choice of one of the two oppositely-oriented, and thus oppositely-signed, versions the so-called *volume element* or *volume form*, the top-degree n-form which is the exterior product of all n basis 1-forms.

This (α, Ω') ordered pair representation of an odd form is cognate to what I call the *(unbound) extremum* representation. It is extremum because the degree of Ω' is n, the *maximum* possible for an n-dimensional manifold. It is unbound because the sense of the orientation of Ω' is *not fixed* and is allowed to vary between the opposites represented by Ω and $-\Omega$ (with a corresponding change in the sign of α).

Instances of cognates to the extremum representation of an odd form occur in various publications. The differential geometric representation of an odd form as the ordered pair (α, Ω) is found in Burke's book [34, pp. 188 f.] and Bossavit's on-line book [23, pp. 67 f.] and compendium [24, pp. 12 f.] (where it is written as $\{\alpha, \Omega\}$). The extremum representation is much more similar to the version $(\alpha, \{\Omega\})$ which appears in Burke's draft paper [36, pp. 4 f.] than it is to the previous two examples. This is because there Burke explicitly defines the use of curly brackets $\{\ \ \}$ in the expression $\{\Omega\}$ to mean the equivalence class $\{\Omega' \mid \Omega' = k\Omega \text{ for } k > 0\}$. Later in this paper, we will need to enlarge this equivalence class by adopting a more general form of its defining property.

This symbolic representation, along with the others discussed in this paper, has a unique pictorial counterpart. Some of these pictorial representations have previously appeared in the literature. I have found images of the extremum representation of odd quantities, but only in Burk's book [34, pp. 188–190].

As good as it is, an extremum representation is still not natural for an odd form, but the *native* representation is. In the native representation of an odd form the two parts of the ordered pair have *complementary degrees*, that is, degrees that sum to n, the maximum degree of a differential form. In systematic taxonomy the native representation is called the *unbound correlated* representation. This representation is correlated because the degrees of the two parts in it are *inversely related*. It is unbound because it does not depend on a specified fixed reference orientation as Ω or $-\Omega$.

The unbound correlated representation of odd forms is cognate to an expression described by Burke in his explanation of his self-named "William's twisted notation" for odd forms. This description occurs in his on-line draft paper [36, pp. 5 f.] as the penultimate step of the conversion of the $(\alpha, \{\Omega\})$ representation of an odd form to William's twisted notation. Unfortunately, his analysis runs through it without recognizing its full significance. It is this expression, only an intermediate step to Burke, which was the crucial inspiration for this work.

While Burke's paper contains the only instance that I have found of a symbolic cognate of the native, or unbound correlated, representation of an odd form, graphical versions of it are common among the works of some authors. The earliest illustrations of the native representation of odd quantities that I have found are in Schouten's works: the German language book [158, p. 22], the paper jointly authored with van Dantzig [164], as well as Schouten's subsequent books [161, p. 28] and [162, pp. 31–33, 55]. They are also found in Burke's book [34, pp. 185–198, 276–282, et al.] and his papers [33, 35, 36], as well as Bossavit's on-line book [23, p. 72] and Jancewicz's papers [105, 106, 108].

Later in this paper we discuss altogether three types of symbolic representations of not only odd forms, but also even forms, and odd and even multivectors. These representations are equivalence classes of ordered pairs with each part of the pair expressed, in general, as a sum of forms or multivectors.

As just reviewed, some of these representations appear in various publications as cognates that are limited to odd forms and multivectors. However, I do not know of any earlier systematic development of all three representations for both odd and even quantities. Furthermore, nowhere else have I found these representations exploited as the basis for an exterior algebra or an exterior calculus as is done here.

Later in this work, when constructing the exterior algebra of the *(bound) correlated* representation we are naturally and necessarily lead to the *orientation congruent* ($\mathcal{OC}$) *algebra*. The orientation congruent algebra is most easily defined, as we do in this paper, by altering the generators and relations definition of Clifford algebra.

The orientation congruent algebra is a *Clifford-like* algebra. Clifford-likeness may be defined by considering products of the basis multivectors that are derived from an orthonormal set of basis vectors. Then the Clifford-likeness of some algebra means that the product of two given basis multivectors in that algebra is same as their Clifford product up to sign. In practice, humans or computer programs that can calculate the Clifford product can also calculate the orientation congruent product by using the explicit Clifford-like formula for the $\mathcal{OC}$ product which is provided later.

In addition to its Clifford-like status, the $\mathcal{OC}$ algebra is also one of a large class of nonassociative algebras known as *noncommutative Jordan* algebras. The Cayley or multiplication table of products of basis multivectors (as defined in the previous paragraph) and their negatives defines a *loop* which appears to have unique properties. However, a thorough investigation of the algebraic properties of the orientation congruent algebra will have to wait until another publication.

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MOSTLY OR ALL DISCARDABLE STUFF

The $\mathcal{OC}$ contraction is key to a theory of *differenced pullbacks* that allows them to both commute with exterior differentiation and maintain sign consistency when applied across forms of all degrees. Thus, we naturally resolve the dilemma that confronted Warnick, Selfridge, and Arnold in their attempt to formulate of these electromagnetic field jump conditions. Even after their reported consultation [195, p. 332, fn.] with Burke, they could manage it only by an arbitrary, ad hoc modification of a standard result.

Warnick, Selfridge, and Arnold introduced the boundary projection operator in their 1994 paper [195] by defining it in terms of what they called the *interior product* of differential forms. Their interior product is also known among Clifford algebraists as the Clifford contraction.

We provide it the end of this paper, using the *contraction operator* of the orientation congruent algebra. The $\mathcal{OC}$ contraction is key to a theory of *differenced pullbacks* that allows them to both commute with exterior differentiation and maintain sign consistency when applied across forms of all degrees. Thus, we naturally resolve the dilemma that confronted Warnick, Selfridge, and Arnold in their attempt to formulate of these electromagnetic field jump conditions. Even after their reported consultation [195, p. 332, fn.] with Burke, they could manage it only by an arbitrary, ad hoc modification of a standard result.

At the end, we apply our work to a problem encountered by Warnick, Selfridge and Arnold [195] when using their *boundary projection operator* method[5] in electrodynamics. In their paper [195, p. 332, fn.] they consult Burke who is also unable to resolve the dilemma they face. Warnick et al. expediently choose to modify Burke's rule $n \wedge \{(\alpha_s, \Omega_s)\} = \{(\alpha, \Omega)\}$ [34, pp. 192 f.]. Burke gives this formula as necessary for pullback to commute with the exterior derivative operator d. The version they use, $\{(\alpha_s, \Omega_s)\} \wedge n = \{(\alpha, \Omega)\}$, does not commute with d, but it does allow them to write the boundary conditions consistently with the same sign for both the odd 1-form D and the odd 2-form H.

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In this work I present the ideal treatment of odd quantities by using a symbolism that fully respects their inherent, two-part, complementary orientation. Using the *bound correlated representation* I provide a new formulation of the exterior algebra and calculus that treats both odd and even differential forms in a uniform way. These developments require the definition of the *orientation congruent algebra* and its associated contraction operator. The orientation congruent algebra is easily defined by altering the generators and relations definition of Clifford algebra. The orientation congruent algebra is a *Clifford-like algebra* that is also a member of a large class of nonassociative algebras, the *noncommutative Jordan algebras*. However, a thorough investigation of the algebraic properties of the orientation congruent algebra will have to wait until another publication.

The $\mathcal{OC}$ contraction operator is the key to a theory of *differenced pullbacks* that allows them to both commute with exterior differentiation and maintain sign consistency when applied across forms of all degrees. In later work I plan to apply the $\mathcal{OC}$ contraction to a related sign inconsistency that occurs in the derivation of the parities of the space+time, $(3+1)$-dimensional, electromagnetic field quantities when they are split from their spacetime, 4-dimensional, counterparts [128].

I do not know how much computational advantage is provided by adopting the concepts of this paper. For that reason this work may receive little attention from engineers who are not attuned to conceptual unity and simplicity, or, to use a stuffy word, elegance, but instead are driven by the need to analyze and design physical systems. For them, as pointed out by Bossavit [23, p. 7], numerical results and computational efficiency are paramount.

Theoretical physicists may be more interested, especially if, as I believe it will, the formalism presented here resolves a particular controversy in electrodynamics. This problem stems from the inconsistency between the parities usually assigned to electromagnetic quantities within the continuum of 4-dimensional spacetime and the parities usually assigned to them in an observer-split $(3+1)$-dimensional space+time. It was analyzed by Marmo, Parasecoli, and Tulczyjew in Reference [128] and recapitulated by Marmo and Tulczyjew in Reference [129],

Elsewhere, in relativity theory, there is an apparent inconsistency between the parity of the fundamental electromagnetic quantities as formulated in the 4-dimensional spacetime continuum and as formulated in the $(3+1)$-dimensional space+time split of an observer. This is discussed by Marmo, Parasecoli, and Tulczyjew in Reference [128] and recapitulated by Marmo and Tulczyjew in Reference [129]. The first paper is reviewed as suffering from "some unjustified polemics" by Schmidt in Zentralblatt [157]). This controversy, as well, could be resolved with these new techniques. Unfortunately, I do not have time to do it here.

Also of interest to physicists and physically-minded mathematicians may be the fact that the orientation congruent algebra $\mathcal{OC}3$ properly contains the *hyperbolic quaternions* of Alexander MacFarlane []. MacFarlane's nonassociative algebra of

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[5] This method has also been exploited by R. Bhakthavathsalam as mentioned on his webpage [19].

hyperbolic quaternions can be used to calculate the relative velocities of reference frames in relativity theory. The orientation congruent algebra's role in this must be related to Ungar's use of *gyrogroups* in relativity theory and hyperbolic geometry [], but I have not yet had time to elucidate it.

Mathematicians may be most curious about this last point as well as the orientation congruent algebra's relationship to the nonassociative differential geometry of Sabinin et al. []. The relationship of this nonassociative differential geometry has already been worked out for gyrogroups []. In addition, further combinatorial work is required to define the standard basis of orientation congruent algebras of arbitrary finite dimensions. With the general standard basis nailed down it appears possible to define the meet and join of oriented flats in oriented projective geometry of arbitrary dimensions.

****

Essentially this means that the signs of the terms in the sum of multivectors that is OC product of two given elements may differ from the signs of the corresponding terms in the sum of multivectors that is the Clifford product of the same two elements.

By Clifford-like we mean that the signs of the terms in the OC product of two given elements may differ from those of the terms in the Clifford product of the same two elements. Both products, however, contain the same set of terms considered independently of sign. In addition to its Clifford-like status, the OC algebra is also one of a large class of nonassociative algebras known as *noncommutative Jordan* algebras.

Nor do I know of the prior presentation of this paper's exterior algebra and calculus of odd and even forms both represented in the *bound correlated* format.

Here the word *bracket* does not signify an inner product. Instead, it reflects that the shape of the brackets indicates whether the ordered pair they contain represents a correlated or extremum equivalence class. Bound brackets are bound to one of the two possible choices of oppositely-oriented volume forms, while unbound brackets do not depend on that arbitrary choice. The *binding* of a bracket, whether it is bound or unbound, is indicated by the punctuation separating the two parts of the bracket.

The unbound correlated representation of odd quantities is physically and conceptually natural. Therefore, I give it the special designation *native*.

The unbound extremum bracket representation of odd forms is cognate to the differential geometric representation of an odd form as the ordered pair (α, Ω) found in Burke's book [34, pp. 188 f.] and Bossavit's on-line book [23, pp. 67 f.] and compendium [24, pp. 12 f.] (where it is written as $\{\alpha, \Omega\}$). It is even more related to the version $(\alpha, \{\Omega\})$ which appears in Burke's draft paper [36, pp. 4 f.]. The bound correlated bracket representation of odd forms is cognate to an expression described by Burke in his explanation of his self-named "William's twisted notation" for odd forms. This description occurs in Reference [36, pp. 5 f.] as the penultimate step of the conversion of the $(\alpha, \{\Omega\})$ representation of an odd form to William's twisted notation.

Corresponding to each of these symbolic types is a unique pictorial representation, some of which have previously appeared in the literature. The earliest illustrations of the native representation of odd quantities I have found are found in Schouten's works: the German language book [158, p. 22], the paper jointly authored with van Dantzig [164], as well as Schouten's subsequent books [161, p. 28] and [162, pp. 31–33, 55]. The only images of the unbound extremum representation of odd quantities that I have found are in Burk's book [34, pp. 188–190].

Some of the four pictorial representations of odd p-forms and p-vectors presented here, specifically, the unbound correlated, or native, and the unbound extremum ones, do appear in other author's works. On the symbolic side the unbound extremum representation of odd forms is directly equivalent to the usual differential geometric representation of an odd form as the ordered pair (α, Ω). However, I do not know of any simultaneous development before this paper of all four symbolic and pictorial representations for both odd and even quantities. Nor do I know of the prior presentation of this paper's exterior algebra and calculus of odd and even forms both represented in the *bound correlated* format.

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My work, of course, has its this paper does have its precedents.

However, one immediate inspiration for this work is Burke's self-named "William's twisted notation." The native symbolic representation of odd forms is implicit in the penultimate step of his explanation of this notation. Burke's explanation begins with the usual differential geometric representation of an odd form as (α, Ω) and step-by-step transforms it into William's twisted notation. Understandably, taking the last step of this transformation was essential for his purposes, but in so doing he does not recognize the significance of his penultimate step.

Although the native representation of odd forms has not previously been developed in a symbolic form, it has appeared in the works of Schouten in a graphical form.

Although the *symbolic* native representation of odd forms has not been previously developed, their *graphical* native representations have illustrated Schouten's works. Schouten appears to be the first to publish such illustrations perhaps as early as 1938 in a German language paper []. They definitely appear in a later English paper jointly authored with van Dantzig [164], as well as Schouten's subsequent books [163]

However, the key idea of a distinguishing internally and externally oriented geometric objects seems to first appear in Veblen and Whitehead's *Foundations of Differential Geometry* [, pp. 55 f.] during a discussion of k-cells.

The differential-geometric treatment of orientation just sketched, has been applied to physics by William L. Burke in References [33, pp. 188 f.] and [34, 36], and, has been applied to electrical engineering by Alain Bossavit in References [23, pp. 67–73] and [24, pp. 12 f.].

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Ignorance or misunderstanding of the orientation of physical quantities also leads to unnecessary continuing controversy among engineers and physicists over the nature and classification of electromagnetic quantities. An example is found in the papers of Fournet [73, 74]. Although, as Roche explains in Reference [147], various subtle points of physical interpretation contribute to this confusion, it is, in no small part, due simply to not realizing that half the directed physical quantities of a theory—the odd ones—have a bipartite directional nature. The expression, traditionally favored by physicists, of quantities in tensor analytic form as a *single* set of components allowing certain transformations only hinders our discernment of the directional properties of physical quantities. This is evidenced in another paper [148, p. 194] in which Roche quotes Sommerfeld and Weyl. In both quotations the magnetic intensity H is treated as a planar quantity when it is actually an odd quantity with both a planar direction, its 2-form-derived part, and a linear direction, its 1-form-derived part.

In the book *Classical Electrodynamics* [100, pp. 65 f.], Ingarden and Jamiołkowski discuss the concepts of *intensity quantities* and *magnitude quantities*. The classification of the field quantities E, D, B, and H of electrodynamics in their Table 1 clearly shows that the intensity quantities E and B correspond to even quantities and the magnitude quantities D and H correspond to odd ones. As mentioned by these authors this classification supports the Lorentz-Abraham classification that groups first E and B, and then D and H as analogous fields. Ingarden and Jamiołkowski also discuss the competing Heaviside-Hertz analogy of first E and H, and then D and B. They go on to say that "...all these similarities are of limited importance."

My initial motivation for this work was to symbolically mimic the geometric representations of *odd* (or *twisted*) differential forms and to construct the exterior algebra of both odd and *even* (*ordinary*) differential forms expressed in those representations. The current literature—excluding tensors, abstract or indexed, since they are not pure and simple differential forms—contains only one symbolic and two geometric representations of odd forms.

Actually, one other symbolic representation exists. William L. Burke was an avid popularizer of both odd and even differential forms and their glorious pictures. His draft paper [36, pp. 5 f.] introduced what he called *William's twisted notation* for odd differential forms. The very beginning of my inquiry was the attempt to do exterior algebra in a modified form of William's twisted notation.

We complete that list to a total of six representations: three geometric and their corresponding symbolic forms. One of the new representations corresponds more naturally to the physical origin of odd forms. They all treat *both* even and odd differential forms in a completely parallel way. Along the way we develop an apparently . We formulate an exterior calculus and apply it to the dilemma of .

We symbolically mimic the geometric representations of *odd* (or *twisted*) differential forms. The current literature contains only one symbolic and two geometric representations of odd forms. We complete the list to a total of three pairs of geometric representations and their symbolic analogs. These pairs treat odd and *even* (*ordinary*) differential forms uniformly. The newest one corresponds more naturally to the physical origin of odd forms. We construct the exterior algebra and calculus of odd and even differential forms in these representations. For this, we define a new, Clifford-like, noncommutative Jordan algebra and dub it the *orientation congruent* ($\mathcal{OC}$) algebra. Our *$\mathcal{OC}$ contraction operator* solves the dilemma of ABC over abandoning the standard pullback or treating differential forms of all degrees uniformly.

We symbolically mimic the geometric representations of *odd* (or *twisted*) differential forms, treating the odd and *even* (*ordinary*) ones uniformly, and complete their known representations to three geometric-symbolic pairs. The newest one more naturally parallels the physical origin of odd forms. We construct the exterior algebra and calculus of forms in these representations. For this, we define a new, Clifford-like, noncommutative Jordan algebra, dubbing it the *orientation congruent* ($\mathcal{OC}$) algebra. The $\mathcal{OC}$ *contraction operator* solves the dilemma of Warnick et al. over abandoning the standard pullback or treating differential forms of all degrees uniformly when finding the jump discontinuities of the electromagnetic D and H.

****

First, let us establish some useful terminology. Bernard Jancewicz, in his papers [105, 106], generalized the meaning of the word *directed*, used earlier in a paper of Lounesto, Mikkola, and Vierros [119], by employing it in the blanket term *directed quantities* to embrace all the following: even and odd multivectors, and even and odd multiforms. We also use it that way in here. The word *orientation* is rather overworked in this paper, occurring in many senses and contexts. Therefore, let us call that property of a directed quantity describing whether its orientation is even or odd the *orientation parity* (or *o-parity*) of that quantity.

This is a mathematical study of the of the *odd* (*twisted*) differential forms that are used to model the quantities of the classical field theories of physics. We consider the following questions:

- What are some geometric representations of these odd forms?
- In particular, what is their natural geometric image?
- What abstract spaces correspond to these geometric images?
- How are exterior (Grassmann) algebras defined for these spaces?
- What nonassociative algebra is required to define these exterior algebras?
- What exterior calculus is derived from one of these exterior algebras?
- What are the physical applications, if any, of this exterior calculus?

> My passion for diff'rential forms
> Defies all traditional norms;
> It may make you queasy,
> But it's really quite easy;
> It ought to be taught in the dorms.
>
> Zbigniew Peradzyński [138, p. 145]

As Burke [34, pp. 176, 268 ff., 285 ff.] and Tucker [179]) point out, many, but not all aspects of the physical quantities and properties described by field theories can be expressed in the coordinate free language of exterior calculus and differential forms. In wide areas of physics and engineering the calculus of exterior differential forms has proved an efficient, versatile tool for formulating and analyzing the topological and differential geometric properties of physical theories. Some examples are gravitation [133], electromagnetism [23, 58, 82, 114], and fluid dynamics [137, 138, 146].

However, it is Maxwell's electromagnetic theory, above all other fields of application, that mates with the exterior calculus as hand to glove. Its unique combination of simplicity and comprehensiveness makes electrodynamics *the* physical theory par excellence for exemplifying the exterior calculus.

In 1963 we find evidence of the particular ease of expressing the laws of electrodynamics with differential forms. It occurs in the pioneering popularizing text, Harley Flanders' *Differential Forms with Applications to the Physical Science* [72]. Electromagnetic fields are the subject, on page 16, of the book's first physical example. Then, by page 44, "Maxwell's Field Equations" is the first full section involving a physical application. Later in 1985, Burke wrote in *Applied Differential Geometry* [34, p. 272]:

> There is a natural match between electrodynamics and differential forms, and they do more for electrodynamics than say, for elasticity or fluid dynamics.

We exploit electrodynamics for examples in this paper. In the last Section we discuss an application of the correlated bracket exterior calculus to Maxwell's classical theory of electromagnetism.

****

We need differential forms so that, in a nonmetrical setting, we can combine one directed quantity, such as a vector, with a second one in a measurement operation (*evaluation*) to give a signed number. In this example, the second directed quantity is a covector or linear form. Similarly, we need odd differential forms so that, also in a nonmetrical setting, we can calculate the quantity of something whose "direction" is an arithmetical sign, such as the amount of positive or negative electrical charge that occurs within a volume of space (always positively signed) to also give a signed number. In this case the amount per volume would be represented by an odd 3-form, while the volume of space would be represented by an odd 3-vector.

The first treatments of odd quantities is cloaked are the language and notation of traditional tensor analysis such as Schouten's books and papers. It is unclear to me if de Rham was the first, or, perhaps, one of the first, to introduce a notation and theory of odd differential forms, but we take the term *odd*, the English translation of the original French *impair*, from his 1955 book *Variétés Différentiables* (*Differentiable Manifolds*) [53, pp. 19, 22 ff.].

The numbers of various types of special tensors and terms and associated with them seem to have multiplied with the same fecundity as their indices.

Perhaps only the romantically mysterious attraction of odd quantities can explain why even math grad students, such as Tiee [174], take a stab at explaining them.

For a formal treatment of relative tensors (including tensor densities and pseudotensors) in terms of a group acting on a set see Yokonuma's book [203, pp. 68–63] or the paper of Hidaka, Arima,and Asaeda [99]. A modern abstract tensor-algebraic formulation of relative tensors can be found in Shaw's book [167, pp. 359–362].

Using the following passage paraphrased from (and further details from) Trautman's talk notes [178, p. 3] the approach of Yokonuma's book [203] to tensor densities, and pseudo-tensors can be related to sections of bundles.

Even in more advanced works, such as Wasserman's [] odd orientations are either not treated or given a cursory mention.

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Although electrodynamics is a most complete exemplifier of differential forms, its comprehensiveness may still be enhanced by rewriting Maxwell's equations in symmetrical form so that they include the, as yet unobserved, magnetic monopole. We do not do it here, but if the field quantities arising from a monopole's magnetic charge were added, we would find that their representation requires differential forms in all eight combinations of degree and orientation allowed in three-dimensional space.

Unfortunately, this last property, the orientation of a differential form, is often the subject of a one-sided discussion in both mathematical and physical works.

That is, frequently only forms with an *even* or ordinary orientation are included—those with an *odd* or *twisted* orientation being neglected. So we begin with an introduction to both orientations.

In between, we develop a geometrically inspired, algebraic formalism[6] for the exterior product of both odd and even forms. For its implementation, this formalism requires an apparently new algebra, which we dub the *orientation congruent* (or *OC*) algebra.

The most accessible definition of the orientation congruent algebra is its axiomatic formulation in terms of generators and relations. We present this axiom set here as a modification of the one for Clifford algebra. For calculations it is also useful to define the OC algebra as a *Clifford-like* algebra and we do that also.[7]

By Clifford-like we mean that the signs of the terms in the OC product of two given elements may differ from those of the terms in the Clifford product of the same two elements. Both products, however, contain the same set of terms considered independently of sign. In addition to its Clifford-like status, the OC algebra is also one of a large class of nonassociative algebras known as *noncommutative Jordan* algebras.

At the end, we apply our work to a problem encountered by Warnick, Selfridge and Arnold [195] when using their *boundary projection operator* method[8] in electrodynamics. In their paper [195, p. 332, fn.] they consult Burke who is also unable to resolve the dilemma they face. Warnick et al. expediently choose to modify Burke's rule $n \wedge \{(\alpha_s, \Omega_s)\} = \{(\alpha, \Omega)\}$ [34, pp. 192 f.]. Burke gives this formula as necessary for pullback to commute with the exterior derivative operator d. The version they use, $\{(\alpha_s, \Omega_s)\} \wedge n = \{(\alpha, \Omega)\}$, does not commute with d, but it does allow them to write the boundary conditions consistently with the same sign for both the odd 1-form D and the odd 2-form H.

The main lesson of this paper for differential geometers is that odd differential forms exist and can be defined on an orientable manifold without defining a global volume form (or without using Bossavit's double cover in the case of a nonorientable manifold?).

Some quotes from Brian D. Conrad's differential geometry class notes [46] follow. Page 1: ... the 1-dimensional top exterior power $\bigwedge^n(V)$ (understood to mean F if $n = 0$) is sometimes called the determinant of V, and is denoted $\det(V)$. Page 2: In the special case $E = TX$, the determinant bundle $\det(TX)$ is often called the *orientation bundle* of X; this line bundle is closely related to the theory of orientation on manifolds, as we shall discuss later.

The orientation congruent algebra of an n-dimensional positive-definite vector space, $\mathcal{OC}_n$, is a *rooted hyperbolic space* of dimension $2^n - 1$ plus the dimension of the root 1.

[6]The algebraic developments presented here are motivated by geometric pictures. As such they exemplify the dialectic of right brain imagery and left brain lexical abstraction essential to mathematical research. See Vinogradov's essay in Nestruev [134, pp. 213 f.] or the first couple of pages of Manin's preprint [127]. For comments on the view that visualization has no or even a hindering role in mathematics, see the interview with Pierre Cartier [165, p. 27].

[7]The equivalence of these definitions is wearily proved in my draft paper [56, pp. 20 ff.].

[8]This method has also been exploited by R. Bhakthavathsalam as mentioned in his webpage [19].

The following quote is from the paper [70, p. 15] or [71, p. 15] by Figueroa-O'Farrill et al. which is based on Lawson and Michelsohn's book *Spin Geometry* [117]. Their sign convention $\mathbf{x} \cdot \mathbf{x} = -\|\mathbf{x}\|^2$ is the opposite to the one used in this paper. Cite new format Burke [36].

> If ω is a p-form and $\star\omega$ its Hodge dual, then their Clifford actions are related by
> $$\star\omega = (-1)^{p(p+1)/2}\omega \cdot \boldsymbol{\nu} \qquad (23)$$
> where $\boldsymbol{\nu}$ is the volume form.

2. Directed Quantities: Their Decomposition and Representation

We adopt the convention, found in Darling's book [52, pp. 3, 41 et al.], of using a bullet or filled circle for function composition as in $(g \bullet f)(x) = g(f(x))$. This is because, following the convention found in Rota and Stein's paper [153], we reserve the more commonly used open bullet or circle for the Clifford ("circle") product as in $\mathbf{e}_1 \circ \mathbf{e}_{12} = \mathbf{e}_2$. Since the orientation congruent product is a modified form of the Clifford product, this last convention leads to the denotation of the orientation congruent product as a circled circle as in $\mathbf{e}_1 \circ \mathbf{e}_{12} = -\mathbf{e}_2$

The relationship of a vector to its graphical representation is direct and simple, but that of a linear form is indirect and complicated. Therefore, we start with a vector space and the concomitant multivectors of its exterior algebra.

In this Section we present the decomposition and representation of two prototypical examples, the even vector and the odd bivector. Although the result may seem artificial and superfluous, we have chosen the even vector to be the subject of the first decomposition. On the other hand, the simplicity of this choice recommends it as an introductory example. In any event, the decomposition of an odd bivector is completely natural and the reader should be more comfortable with these decompositions after considering it in Subsection 2.3 titled "The Decomposition and Representation of Odd Bivectors" below

The first subsection's example helps us develop some basic geometric intuition and fundamental symbolic machinery. Although it is applied there to only the *native* representation of even multivectors, in later subsections this framework allows us to construct the other two *correlated* and *extremum* representations for general directed quantities.

2.1. The Decomposition and Representation of Even Vectors.

In introductory courses a vector is commonly defined as a quantity with a *magnitude* (its length) and a *direction*, and then illustrated as a line segment with an arrowhead at one end. This definition is, of course, a metrical one. It suggests the multiplicative decomposition of a nonzero, Euclidean vector $\mathbf{b}$ into $b\hat{\mathbf{b}}$, the product of the real number b, the magnitude or *norm* of $\mathbf{b}$, defined with the help of a scalar product as $b = \|\mathbf{b}\| := \sqrt{\mathbf{b} \cdot \mathbf{b}}$, and the unit vector written with a circumflex as $\hat{\mathbf{b}} := b^{-1}\mathbf{b}$ giving the direction of $\mathbf{b}$.

In their 1989 paper [119, p. 100] Lounesto, Mikkola, and Vierros introduced a refinement of this decomposition by further dividing the direction of a vector into an *attitude* and *orientation*. For these authors a vector's attitude is the line containing it and its orientation is the sense in which the vector points along that line. Lounesto et al. also applied this three-part decomposition to simple bivectors so that a bivector's attitude is the plane containing it and its orientation is the sense of its "rotation" in that plane.

Working as they did with geometric algebra, that is, Clifford algebra interpreted geometrically, Lounesto et al. had at hand a scalar product (or more generally a covariant metric tensor) with which to find the magnitude of a vector, bivector, or general p-vector. However, we want a similar, but basis- and metric-free dissection of a vector. We find that the finest, basis-independent and nonmetric decomposition of a vector is also tripartite but redundant. Its geometric version is given in Figure 2.1.

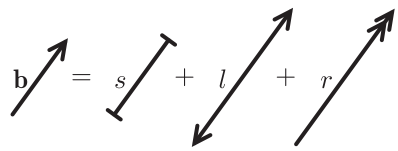


Figure 2.1. This is the finest geometric, basis-independent and nonmetric decomposition of a nonzero vector $\mathbf{b}$. It consists of a line segment s, a line l, and a ray r. The ray r is drawn with a double arrowhead to distinguish it from the vector $\mathbf{b}$. The equal and addition signs are metaphorical.

The finest geometric, basis-independent and nonmetric decomposition of a nonzero vector $\mathbf{b}$ is tripartite:

(1) a line segment s parallel to $\mathbf{b}$, and which would exactly overlap $\mathbf{b}$, carrying the vector's *attitude* and its *relative magnitude*, the nonmetric equivalent of the vector's magnitude;
(2) a line l parallel to $\mathbf{b}$ carrying purely the vector's *attitude*; and
(3) a ray r parallel to $\mathbf{b}$ carrying the vector's *attitude* and its *orientation*, that is, the sense of its direction.

We seek symbolic formulations that mimic or are suggested by the three parts of the geometric, basis- and metric-independent decomposition of a vector shown in Figure 2.1. Although, when appropriate we look for metric equivalents as well. We begin this quest by examining Jancewicz's definition of a numerical relative magnitude for a vector.

When a scalar product is not available, we cannot calculate the magnitude of a vector or, more generally, a multivector. In his papers [105, pp. 389 f.] and [106, pp. 227 f.] Jancewicz considers nonmetric spaces. He cites the work of Lounesto et al. [119] and points out that two vectors $\mathbf{a}$ and $\mathbf{b}$ have the same *attitude* if they are parallel, that is, if $\mathbf{a}$ and $\mathbf{b}$ are such that there exists a real scalar $\lambda \in \mathbb{R}$ with $\mathbf{b} = \lambda\mathbf{a}$. Nevertheless, Jancewicz states that, by using the last equation, we can always define a [unique] *relative magnitude* for [nonzero] parallel vectors as $|\lambda|$. In full, he says that $|\lambda|$ is the *magnitude of* $\mathbf{b}$ *relative to* $\mathbf{a}$.

We notice two things about the relative magnitude of $\mathbf{b}$. First, λ is invariant under the sign change $-\mathbf{b}$; this corresponds to the relative magnitude being associated geometrically with an *unoriented* line segment. Second, the relative magnitude of $\mathbf{b}$ requires a vector parallel to $\mathbf{b}$, implicit or explicit, for its definition. In the geometric interpretation of relative magnitude this fact corresponds to requiring the line segment s to be parallel to $\mathbf{b}$. Thus, in the nonmetric case the relative magnitude of $\mathbf{b}$ is separable from its attitude in only a conceptual or potential (if a metric is introduced), not an actual, sense.

In the metric case, however, we can define the relative magnitude's counterpart as $\|\mathbf{b}\|$, the magnitude of $\mathbf{b}$, without direct reference to the vector's attitude. The attitude of $\mathbf{b}$ still plays a role in determining $\|\mathbf{b}\|$, which is especially evident if the metric varies with direction. However, once we define or calculate the magnitude of $\mathbf{b}$ we are done with its attitude and need not refer to it again. In other words, the magnitude is a *scalar*, cleanly divorced from any vectorial properties such as attitude. In general, though, we work without a metric.

We now provide a symbolic formulation, different than the relative magnitude of a vector $\mathbf{b}$, that is the counterpart of the geometric, nonmetric representation given by the line segment s in Figure 2.1. We choose a representation as a set of vectors. Such a representation is similar to the set $\ker \alpha$ which is used to symbolically capture the geometrical representation of a linear form (covector) α as a pair of oriented planes (see Subsection 3.1 below). We define this nonmetric, symbolic representation first, followed immediately by the analogous metric one.

Definition 2.1. We call the symbolic, nonmetric representation of the line segment s in the decomposition of the vector $\mathbf{b}$ the *weight* of $\mathbf{b}$ and write it in operator notation as $\operatorname{wt}\mathbf{b}$. We define it as the following set:

$$\operatorname{wt}\mathbf{b} := \{\mathbf{v} \mid \mathbf{v} = \pm\mathbf{b}\}. \tag{2.1}$$

Remark 2.2. The weight of a vector $\mathbf{b}$ can be imaged as a separated *tail-centered* or superimposed *midpoint-centered* pair of vectors $\mathbf{b}$ and $-\mathbf{b}$. In Figure 2.1 we have chosen the midpoint-centered representation of $\operatorname{wt}\mathbf{b}$ with short perpendicular slashes at its endpoints. We do not represent $\operatorname{wt}\mathbf{b}$ as a line segment with an outward-pointing arrowhead at each of its endpoints because that is the usual representation of a line.

Definition 2.3. Let V be a metric vector space with a scalar product written as an infixed, centered dot and let $\mathbf{b} \in V$ be a nonzero vector. We call the metric counterpart of $\operatorname{wt}\mathbf{b}$ in Definition 2.1 the *metric weight* of $\mathbf{b}$. We write it also in operator notation as $\operatorname{mwt}\mathbf{b}$ and define it as the following set:

$$\operatorname{mwt}\mathbf{b} := \{\mathbf{v} \in V \mid \mathbf{v} \neq 0 \text{ and } \mathbf{v}\cdot\mathbf{v} = \mathbf{b}\cdot\mathbf{b}\}. \tag{2.2}$$

Remark 2.4. In Euclidean spaces the metric weight of a vector $\mathbf{b}$ can be imaged as a *tail-centered* or *midpoint-centered* n-dimensional "sphere" of vectors with the same length as $\mathbf{b}$ and pointing in all directions.

Remark 2.5. We see that both the nonmetric and metric weights of a vector $\mathbf{b}$ are invariant under its inversion: $\operatorname{wt}\mathbf{b} = \operatorname{wt}\pm\mathbf{b}$ and $\operatorname{mwt}\mathbf{b} = \operatorname{mwt}\pm\mathbf{b}$.

The line l, parallel to $\mathbf{b}$ in Figure 2.1, is the geometric representation of the attitude of the vector $\mathbf{b}$. We next provide the symbolic counterpart of l which we call simply the *attitude* of $\mathbf{b}$. Since the attitude employs only the nonmetric vector space relationship of being parallel, there is no metric version of the attitude. Therefore, we immediately state the following set-of-vectors definition of the attitude.

Definition 2.6. Let $\mathbb{R}^\bullet$ be the set of real numbers excluding zero and let V be a vector space containing $\mathbf{b}$. We call the symbolic, nonmetric representation of the line l in the decomposition of the vector $\mathbf{b}$ the *attitude* of $\mathbf{b}$ and write it in operator notation as $\operatorname{att}\mathbf{b}$. We define it as the following set:

$$\operatorname{att}\mathbf{b} := \{\mathbf{v} \in V \mid \mathbf{v} = a\mathbf{b} \text{ and } a \in \mathbb{R}^\bullet\}. \tag{2.3}$$

Remark 2.7. There is no zero vector in $\operatorname{att}\mathbf{b}$. Therefore the set of points at the tips of all the vectors in $\operatorname{att}\mathbf{b}$ is not quite a line because one point, the origin, is missing. It is, however, a *punctured* line. Strictly, we should have indicated this in Figure 2.1 by placing an open dot, the conventional representation of a missing point, at the midpoint of the line l.

Remark 2.8. If $\mathbf{b}$ is nonzero, the set $\operatorname{att}\mathbf{b}$ actually defines an equivalence class of vectors in $V^\bullet := V \setminus \{0\}$. In projective geometry, where it is interpreted as a projective point, this set is also called a *ray*.

The ray r, parallel to $\mathbf{b}$ in Figure 2.1, is the geometric representation of the direction of the vector $\mathbf{b}$. We next provide the symbolic counterpart of r which we call simply the *direction* of $\mathbf{b}$. Unlike the attitude, although the direction employs only the nonmetric relationship of being parallel, the direction has a metric generalization. This generalization is the *metric orientation*. In turn, it leads by analogy to a nonmetric version, the *extrinsic orientation*. We define these two kinds of orientation later in this subsection. But first we state the following set-of-vectors definition of the direction.

Definition 2.9. Let $\mathbb{R}^+$ be the set of positive real numbers and V be a vector space containing $\mathbf{b}$. We call the symbolic, nonmetric representation of the ray r in the decomposition of the vector $\mathbf{b}$ the *direction* of $\mathbf{b}$ and write it in operator notation as $\operatorname{dir}\mathbf{b}$. We define it as the following set:

$$\operatorname{dir}\mathbf{b} := \{\mathbf{v} \in V \mid \mathbf{v} = a\mathbf{b} \text{ and } a \in \mathbb{R}^+\}. \tag{2.4}$$

Remark 2.10. If $\mathbf{b}$ is nonzero, the set $\operatorname{dir}\mathbf{b}$ is actually an equivalence class in $V^\bullet$. Although, it defines an *open* ray without an endpoint, the set $\operatorname{dir}\mathbf{b}$ can be identified with the (closed) ray r of Figure 2.1. We briefly mention that this set is interpreted in *oriented projective geometry* as an (oriented) point. Here are some references for its application in computational geometry, computer graphics, computer vision, and robotics: Stolfi's book [173] based on his dissertation [172], Stolfi's extended abstract [171], Kirby's article [112], Lazebnik's thesis [118, pp. 15 ff.], and Mason's book [130, ch. 5]. Choi uses oriented projective geometry in a paper [45, pp. 72, 74] on geometrical structures and Coxeter groups. For a brief mathematical description of this geometry as a double covering or fibration see the paper of Below, Krummeck, and Richter-Gebert [17, p. 6 of preprint].

Our analysis of the intrinsic, nonmetric decomposition of a vector is complete. Yet the descriptions of the geometric ray r and its symbolic counterpart, the direction $\operatorname{dir}\mathbf{b}$, suggest the possible existence of another, more general concept, the *orientation* of the vector $\mathbf{b}$, that is not bound to the attitude of $\mathbf{b}$. At first glance it appears that in an intrinsic, nonmetric setting a vector's orientation is not separable from its attitude. Yet our experience with the relative magnitude, weight, and metric weight of a vector shows that introducing a metric does allow movement away from the attitude of $\mathbf{b}$. Thus, we first define the metric orientation of a vector, and then, by analogy, its nonmetric orientation.

The extrinsic orientation of a vector (or of even quantities in general) may seem somewhat artificial. However, the orientation of odd quantities is inherently extrinsic. Later we will see that it is only by using the extrinsic orientation of even quantities that we can treat both odd and even quantities in the same symbolic system. We look next at an example that illustrates how the introduction of a metric allows us to generalize the direction $\operatorname{dir}\mathbf{b}$ and thus free it from the attitude $\operatorname{att}\mathbf{b}$.

In three-dimensional Euclidean space there exists a plane P perpendicular to a given nonzero vector $\mathbf{b}$. If we add any vector $\mathbf{p}$ in this plane to any positive scalar multiple of $\mathbf{b}$ we obtain another vector $\mathbf{d}$ representing the orientation of $\mathbf{b}$. This vector $\mathbf{d}$ is on the same side of P as is $\mathbf{b}$. Having found a way to move off the line l and outside the set $\operatorname{att}\mathbf{b}$, we can now state a definition.

Definition 2.11. Let V be a metric vector space with a scalar product written as an infixed, centered dot. The metric generalization of $\operatorname{dir}\mathbf{b}$ in Definition 2.9 for any nonzero vector $\mathbf{b} \in V$ is called the *metric orientation* of $\mathbf{b}$. We write it also in operator notation as $\operatorname{mor}\mathbf{b}$ and define it as the following set:

$$\operatorname{mor}\mathbf{b} := \{\mathbf{v} \in V \mid \mathbf{v} \neq 0 \text{ and } \mathbf{v} \cdot \mathbf{b} = a\mathbf{b} \cdot \mathbf{b} \text{ for some } a \in \mathbb{R}^+\}. \tag{2.5}$$

Remark 2.12. If the metric is Euclidean this definition reduces to the simple statement $\operatorname{mor}\mathbf{b} = \{\mathbf{v} \in V \mid \mathbf{v} \cdot \mathbf{b} > 0\}$.

Remark 2.13. The metric orientation is invariant under the transformation $\mathbf{b} \mapsto a\mathbf{b} + \mathbf{p}$ for any $a \in \mathbb{R}^+$ and any $\mathbf{p} \in V$ such that $\mathbf{p} \cdot \mathbf{b} = 0$.

Consider how to convert Definition 2.11 into a definition of a nonmetric, extrinsic orientation. Without a metric we no longer have a perpendicular space, but, if V is n-dimensional, an arbitrary $(n-1)$-dimensional vector space that does not contain a nonzero scalar multiple of $\mathbf{b}$ would work just as well. The next definition formalizes this notion as the extrinsic orientation of $\mathbf{b}$ relative to such a space.

Definition 2.14. Let $\mathbf{b}$ be a nonzero vector in the n-dimensional vector space V. And let W be any $(n-1)$-dimensional subspace of V such that there exists a (unique up to a scalar multiple) covector or linear form β of the dual space V^* that satisfies the conditions $\beta(\mathbf{b}) \neq 0$ and $W = \ker\beta := \{\mathbf{v} \in V \mid \beta(\mathbf{v}) = 0\}$. Then the partial nonmetric analogue of $\operatorname{mor}\mathbf{b}$ in Definition 2.11 for the vector $\mathbf{b}$ is called the *extrinsic orientation of* $\mathbf{b}$ *relative to* W. We write it in operator notation as $\operatorname{eor}(\mathbf{b}, W)$ and define it as the following set:

$$\operatorname{eor}(\mathbf{b}, W) := \{\mathbf{v} \in V \mid \beta(\mathbf{v}) = a\beta(\mathbf{b}) \text{ for some } a \in \mathbb{R}^+\}. \tag{2.6}$$

Remark 2.15. The extrinsic orientation of $\mathbf{b}$ relative to W is invariant under the transformation $\mathbf{b} \mapsto a\mathbf{b} + \mathbf{w}$ for any $a \in \mathbb{R}^+$ and any $\mathbf{w} \in W$.

Next we remove the extrinsic orientation's dependence on a particular, but arbitrary, complementary subspace W by taking the union over all such subspaces.

Definition 2.16. Let $\mathbf{b}$ be a nonzero vector in the n-dimensional vector space V. And let $\mathscr{W}$ be the union of all the W, that is, the $(n-1)$-dimensional subspaces of V described in Definition 2.14 above. Then the complete nonmetric analogue of $\operatorname{mor}\mathbf{b}$ in Definition 2.11 for the vector $\mathbf{b}$ is called the *extrinsic orientation of* $\mathbf{b}$ *in* V. We write it in operator notation as $\operatorname{eor}(\mathbf{b})$ with V understood from the context and define it as the following set:

$$\operatorname{eor}(\mathbf{b}) := \{\mathbf{v} \in V \mid \mathbf{v} = a\mathbf{b} + \mathbf{w} \text{ for some } a \in \mathbb{R}^+ \text{ and some } \mathbf{w} \in \mathscr{W}\}. \tag{2.7}$$

Remark 2.17. The direction of the vector $\mathbf{b}$ $\operatorname{dir}\mathbf{b}$ may be viewed as the nonmetric, intrinsic analogue of the extrinsic orientation of $\mathbf{b}$.

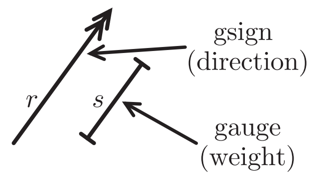


FIGURE 2.2. This is an even vector **b** in its native (bipartite) geometric representation with its *gsign* (*generalized sign*) and *gauge* labeled.

Everything and more is now at hand with which to construct the first of three standard bipartite geometric and symbolic representations of even directed quantities. We call the first geometric one the *native* geometric representation. It is illustrated for the even vector **b** in Figure 2.2. This figure shows an ordered pair of images (r, s) consisting of the ray r, representing the direction $\operatorname{dir}\mathbf{b}$, and the line segment s, representing the weight $\operatorname{wt}\mathbf{b}$.

The symbolic counterpart of the ordered pair of images (r, s) is the *native bracket* $[\![\operatorname{dir}\mathbf{b}, \operatorname{wt}\mathbf{b}]\!]$ written with double left and right square brackets. These brackets represent an equivalence class that is an element in the bipartite Cartesian product space $\mathfrak{OP}(V) \times \mathfrak{W}(V)$ where $\mathfrak{OP}(V)$ is the oriented-projective space of the vector space V and $\mathfrak{W}(V)$ is its weight space. The native bracket is roughly a kind of nonmetric, *geometrically-generalized polar decomposition*. Just as is true for the polar form of a complex number, the native bracket is good for the multiplication, but not the addition, of directed quantities.

Let us now use the same symbols that we employed for the ray r and the line segment s with the new meanings $r = \operatorname{dir}\mathbf{b}$ and $s = \operatorname{wt}\mathbf{b}$. Then from the properties already established for the direction and the weight of an even vector (which are generalizable to all directed quantities) we immediately have the following equalities:

$$
\begin{aligned}
&[\![r, s]\!] = [\![ar, s]\!] \quad \text{for all } a \in \mathbb{R}^+ \text{ and} \\
&[\![r, s]\!] = [\![r, \pm s]\!].
\end{aligned} \tag{2.8}
$$

We also require these equivalence classes to obey some additional sign equalities that are expressed in the following equations:

$$
\begin{aligned}
[\![r, s]\!] &= -[\![-r, s]\!] \quad \text{and} \\
-[\![r, s]\!] &= [\![-r, s]\!].
\end{aligned} \tag{2.9}
$$

From equations (2.8) and (2.9) we see that we have equality of native brackets that have the same contents (disregarding sign) and the same sign parity counting the sign in front of the bracket and the sign of the first (direction) part, but ignoring the sign of the second (weight) part.

Later we develop a total of three standard geometric representations and three corresponding bracket forms for directed quantities. We need some more suggestive, generic terms, less awkward than "the first part" and "the second part" that we can apply to any of the three bracket types. Let us call the first part of any of these brackets, the *gsign*, and the second part, the *gauge*. The reader may take *gsign* as short for *generalized sign* or *geometric sign*, whichever is preferred, as both are apt descriptions of its role in all three brackets. The name *gauge* is also appropriate because in all three brackets it acts as a generalized weight, measure, or magnitude.

These terms also suggest an analogy of the position and function of the gsign and gauge in bracket notation to the position and function of the sign and magnitude of signed numbers. That is, we write the gsign as the first part of a bracket just as we write the number -5 with the sign in front. The second position contains the gauge expressing a weight or measure just as the numeral 5 does. The role of the quantities in this analogy, but not their order, is also consistent with their previous analogy to the polar form of a complex number.

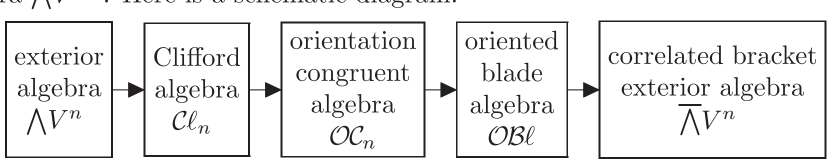
Along with this nomenclatures we also create some notation: the selection operators gsn and gau. From the discussion just above they correspond to the signum operator $\operatorname{sgn} x$ and absolute value operator $|x|$ for any $x \in \mathbb{R}$ (notwithstanding that the absolute value is written as enclosing bars rather than a prefix). The *gsign operator* gsn returns the first part of any bracket and the *gauge operator* gau returns the second part as illustrated here by native brackets:

$$\begin{aligned} \operatorname{gsn}[\![s, r]\!] &= s \quad \text{for the gsign operator and} \\ \operatorname{gau}[\![s, r]\!] &= r \quad \text{for the gauge operator.} \end{aligned}$$

Because surely the native, most natural, form of a vector is just the vector itself, it is only if we demand a bipartite decomposition that the term *native* is a reasonable description for these geometric and symbolic representations of an even vector. However, in the next Subsection we see that odd quantities are naturally bipartite. Therefore, any kind of native form for both even and odd quantities, and calculations among them, must be expanded to their "greatest common multiple"—a bipartition. Thus, we cater to the odd and twisted.

We have learned as much as we can by considering even geometric objects alone. In the following subsection, we look at the odd bivector with its *outer* or *transverse orientation*. Odd quantities are defined, in general, nonmetrically with the help of the extrinsic relative orientation eor and the relation of being not parallel. The next Subsection develops the native representation of the odd bivector and the remaining two standard bipartite representations of directed quantities.

2.2. **The Exterior Algebra of Multivectors.** Multivectors are the visible contents of our brackets. We take their exterior algebra $\bigwedge V^n$ as the substratum of the Clifford algebra $\mathcal{C}\ell_n$. The Clifford algebra is, in turn, a basis for the Clifford-like, orientation congruent algebra $\mathcal{OC}_n$. Then, the orientation congruent algebra is the foundation for the oriented blade $\mathcal{OB}\ell$ algebra. Finally, the oriented blade algebra together with ordinary exterior algebra defines, the correlated bracket exterior algebra $\overline{\bigwedge} V^{p,q}$. Here is a schematic diagram.



2.3. **The Decomposition and Representation of Odd Bivectors.** This Subsection discusses the crucial case of an odd bivector. The initial motivation for this paper was to find a way to symbolically mimic the geometric representations of odd

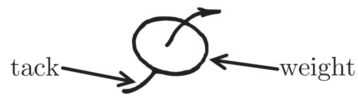


Figure 2.3. This is the native geometric representation of an odd bivector. It was copied (sans labels) from Schouten's book [162, p. 55, Fig. 13].

quantities and construct an exterior algebra of all directed quantities expressed in this form.

As mentioned at the end of the last Subsection there are three such geometric representations. Two of these three, the *native* and the *extremum* one, are already pictured in the literature, but not specifically named. The other, the *correlated* one, appears novel. The native form is native to odd quantities only. None is truly native to even quantities, but the extremum form comes closest. (We cater to the odd and twisted.) Odd multivectors and multiforms have a native representation because they can be directly measured in that form as the physical quantities of theories such as electrodynamics. An explanation for the terms *correlated* and *extremum* will be given later.

We adopt the tilde notation for twisted quantities, as in the expression $\widetilde{D}$ for the twisted 2-form representing the electric displacement vector, that Burke used in his paper [33] and Bossavit adopted later in his on-line monograph [23, pp. 73, 89].

Consider the odd bivector, call it $\widetilde{D}$, shown in Figure 2.3. This and similar images occur in the works of Schouten and van Dantzig [164, pp. 451, 455], Schouten [160, p. 28], [162, pp. 31–33, 55], and Burke [35]. This particular image is copied from Schouten's *Tensor Analysis for Physicists* [162, p. 55].

The reader may have no previous experience with such an object. Therefore we construct it from scratch based on this picture. The image of Figure 2.3 is a *natural* or *native* geometric representation of $\widetilde{D}$; we call its symbolic counterpart $[\![\mathrm{eor}(\pm\,\mathrm{att}*\widetilde{D}, \mathrm{att}\,\widetilde{D}), \mathrm{wt}\,\widetilde{D}]\!]$ THIS SHOULD BE RATHER $[\![\mathrm{gsn}\,\widetilde{D}, \mathrm{gau}\,\widetilde{D}]\!]$ a *native bracket*. We have used double brackets here because later we introduce two more kinds of brackets in the forms $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ and $(\!(\cdot,\cdot)\!)$.

The differential, dual counterpart to an odd bivector, an odd differential 2-form can be integrated to find the surface area of a Möbius strip.

Transverse

There is an analogy to rational numbers and fraction bar notation. We write an even or odd decomposable form as a pair of forms enclosed in a bracket denoting equivalence classes of those pairs. This *correlated grade bracket* (CGB), splits into a *geometric sign* (GS) form and an *oriented measure* (OM) form. The GS form itself represents an equivalence class of ray and vector subspaces associated with semi-oriented projective spaces. The OM form is just an ordinary form.

USE THIS: General terms for the brackets position one and position two will be GS, gsn, geometric sign, and GA, gau, gauge.

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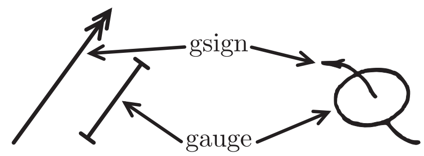


(a)
even vector
native representation

(b)
odd bivector
native representation

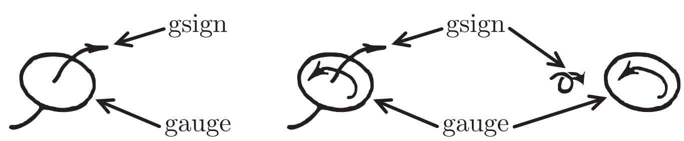
Figure 2.4. These are the native geometric representations of an even vector and an odd bivector with their *gsigns* (*generalized signs*) and *gauges* labeled. The odd bivector was copied (sans labels) from Schouten's book [162, p. 55, Fig. 13].

(a)
native
representation

(b)
correlated
representation

(c)
extremum
representation

Figure 2.5. These are the three standard geometric representations of an odd bivector. These images (sans labels) were copied and modified from Schouten's book [162, p. 55, Fig. 13].



(a)
native
representation

(b)
correlated
representation

(c)
extremum
representation

Figure 2.6. These are the three standard geometric representations of an odd bivector with their *gsigns* (*generalized signs*) and *gauges* labeled. These images (sans labels) were copied and modified from Schouten's book [162, p. 55, Fig. 13].

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2.4. **Basic Terminology.** In the tradition of multilinear algebra a *multivector* of the exterior algebra can always be written as a sum of exterior products with each product containing, say, p vectors as multiplicands. Such a multivector is also called a *p-vector* or a multivector *of degree p*. For convenience, scalars are also called

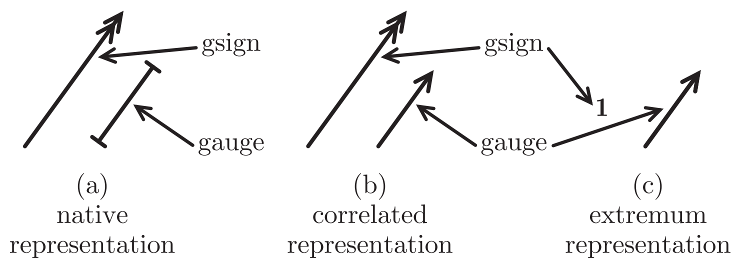


FIGURE 2.7. These are the three standard geometric representations of an even vector with their *gsigns* (*generalized signs*) and *gauges* labeled. Two gsigns are drawn with double arrowheads to distinguish them as rays from the two vectors that occur as gauges.

0-*vectors* and vectors, 1-*vectors*. However, in this paper we practice the custom of *geometric algebra* as found, for example, in the books of Hestenes [91], Hestenes and Sobczyk [97], and Lounesto [123] by calling a p-vector a *homogeneous* multivector (of degree p). Also, as these authors do, we redefine the term *multivector* as a general sum of homogeneous multivectors which are not necessarily of the same degree.

As remarked in the Introduction differential, or linear, forms come in exactly two orientations. The well-known one is the *even* or *straight* orientation. The second, less commonly known one is the *odd* or *twisted* orientation. Several other terms are used in the literature to distinguish these two possible orientations of p-forms (and p-vectors), but in this paper we prefer to use any of the four words just mentioned.

The number of terms applied to differentiate the orientations of these objects may present some difficulty, which is only compounded when we encounter their tensor analytic representation later. Table 2.1 gathers some of the more common names for p-forms of both orientations together with some references in which they appear.

Names for forms are separated by their orientations into the first two columns of Table 2.1. To avoid further confusion we have simply labeled these "Column 1" and "Column 2." Each row presents pairs of complementary names that are used together to distinguish the two orientations. For emphasis the distinguishing terms are printed in boldface. Terms enclosed in parentheses tend to be optional. Authors sometimes substitute a generic word such as *ordinary* for the distinguishing terms of Column 1. Occasionally they write in "diagonal" nomenclature by mixing the Column 1 terms of some row in Table 2.1 with the Column 2 terms of another row.

The French terms in the top row of Table 2.1 seem to have been used first by de Rham. Although the specific reference cited here, his monograph *Variétés Différentiables* [53], may not be the source of their first appearance. The French is used too by some authors writing in English. In the second row we find the translations of de Rham's original French terms as they appeared in the English version of de Rham's book [54]. In the English translation we also find the slight variation of appending the distinguishing phrase *of even* (*odd*) *type* to the description of a p-form.

Testing some Bossavit References [24], [22] and [21].

Table 2.1. Equivalent Terms for the Orientation Parities of Directed Quantities

Complementary Orientation Parities		References
pair	*impair*	de Rham [53, pp. 19, 22 ff.]
even	odd	de Rham [54, pp. 17, 19 ff.], Jancewicz [108]
(ordinary)	twisted	Frankel [75, pp. XX ff.], Burke [33], [34, pp. 151 ff., 183 ff.], Bossavit [21, pp. 67 ff.]
untwisted	twisted	Warnick et al. [195],
(true)	pseudo-	Jancewicz [105, 106], Frankel [76, pp. 86 f.]
polar	axial	Sorkin [170]

Table 2.2. The Gsign and Gauge Types of the Three Representations of Directed Quantities

	Representation		
	Native	Correlated	Extremum
Gsign Type	correlated	correlated	extremum
Gauge Type	weighted	vectorial	vectorial

Table 2.3. Classification of the Three Representations of Directed Quantities by Bigrade Type, Sign Count, and $\mathbf{\Omega}$-Binding

Bigrade	Sign Count		$\mathbf{\Omega}$ -Binding	
	Monosigned	Bisigned	Unbound	Bound
Correlated	$[\![\cdot,\cdot]\!]$ native	$\langle\!\langle\cdot,\cdot\rangle\!\rangle$ correlated	$[\![\cdot,\cdot]\!]$ native	$\langle\!\langle\cdot,\cdot\rangle\!\rangle$ correlated
Extremum	—	$(\!(\cdot,\cdot)\!)$ extremum	$(\!(\cdot,\cdot)\!)$ extremum	—

In Table 4.2 the multivectors in the "Differential Geometry" column are written in multi-index form logically derived from a certain notation for the basis vectors of the tangent space defined locally at a point in a manifold by a particular coordinate chart. Our use of this basis vector notation follows Bossavit in his treatise [23, p. 50]: the notation ∂_i replaces the more traditional form $\frac{\partial}{\partial_i}$.

2.5. **The O-Projective-Linear Space and Native Brackets.** The o-projective-linear bispace

TABLE 2.4. Examples of Directed Multivectors in Six Representations

	Diff. Geom.	Extnd. Grass. Algebra	William's Twisted Notation	Brackets		
				Native	Correlated	Extremum
Even	1	1	1	$\llbracket 1, 1 \rrbracket$	$\langle\!\langle 1, 1 \rangle\!\rangle$	$(\!(1, 1)\!)$
	$\mathbf{e}_1$	$\mathbf{e}_1$	$\mathbf{e}_1$	$\llbracket \mathbf{e}_1, \mathbf{e}_1 \rrbracket$	$\langle\!\langle \mathbf{e}_1, \mathbf{e}_1 \rangle\!\rangle$	$(\!(1, \mathbf{e}_1)\!)$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$\mathbf{e}_2$	$\llbracket \mathbf{e}_2, \mathbf{e}_2 \rrbracket$	$\langle\!\langle \mathbf{e}_2, \mathbf{e}_2 \rangle\!\rangle$	$(\!(1, \mathbf{e}_2)\!)$
	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\llbracket \mathbf{e}_{12}, \mathbf{e}_{12} \rrbracket$	$\langle\!\langle \mathbf{e}_{12}, \mathbf{e}_{12} \rangle\!\rangle$	$(\!(1, \mathbf{e}_{12})\!)$
	$\mathbf{e}_{13}$	$\mathbf{e}_{13}$	$\mathbf{e}_{13}$	$-\llbracket \mathbf{e}_{31}, \mathbf{e}_{31} \rrbracket$	$-\langle\!\langle \mathbf{e}_{31}, \mathbf{e}_{31} \rangle\!\rangle$	$-(\!(1, \mathbf{e}_{31})\!)$
	$\boldsymbol{\Omega}$	$\mathbf{e}_{123}$	$\boldsymbol{\Omega}$	$\llbracket \boldsymbol{\Omega}, \boldsymbol{\Omega} \rrbracket$	$\langle\!\langle \boldsymbol{\Omega}, \boldsymbol{\Omega} \rangle\!\rangle$	$(\!(1, \boldsymbol{\Omega})\!)$
Odd	$(1, \boldsymbol{\Omega})$	r	$\widehat{\boldsymbol{\Omega}}$	$\llbracket \boldsymbol{\Omega}, 1 \rrbracket$	$\langle\!\langle \boldsymbol{\Omega}, 1 \rangle\!\rangle$	$(\!(\boldsymbol{\Omega}, 1)\!)$
	$(\mathbf{e}_1, \boldsymbol{\Omega})$	$r\mathbf{e}_1$	$\widehat{\mathbf{e}}_{23}$	$\llbracket \mathbf{e}_{23}, \mathbf{e}_1 \rrbracket$	$\langle\!\langle \mathbf{e}_{23}, \mathbf{e}_1 \rangle\!\rangle$	$(\!(\boldsymbol{\Omega}, \mathbf{e}_1)\!)$
	$(\mathbf{e}_2, \boldsymbol{\Omega})$	$r\mathbf{e}_2$	$\widehat{\mathbf{e}}_{31}$	$\llbracket \mathbf{e}_{31}, \mathbf{e}_2 \rrbracket$	$\langle\!\langle \mathbf{e}_{31}, \mathbf{e}_2 \rangle\!\rangle$	$(\!(\boldsymbol{\Omega}, \mathbf{e}_2)\!)$
	$(\mathbf{e}_{12}, \boldsymbol{\Omega})$	$r\mathbf{e}_{12}$	$\widehat{\mathbf{e}}_3$	$\llbracket \mathbf{e}_3, \mathbf{e}_{12} \rrbracket$	$\langle\!\langle \mathbf{e}_3, \mathbf{e}_{12} \rangle\!\rangle$	$(\!(\boldsymbol{\Omega}, \mathbf{e}_{12})\!)$
	$(\mathbf{e}_{13}, \boldsymbol{\Omega})$	$r\mathbf{e}_{13}$	$-\widehat{\mathbf{e}}_2$	$-\llbracket \mathbf{e}_2, \mathbf{e}_{31} \rrbracket$	$-\langle\!\langle \mathbf{e}_2, \mathbf{e}_{31} \rangle\!\rangle$	$-(\!(\boldsymbol{\Omega}, \mathbf{e}_{31})\!)$
	$(\boldsymbol{\Omega}, \boldsymbol{\Omega})$	$r\mathbf{e}_{123}$	$\widehat{1}$	$\llbracket 1, \boldsymbol{\Omega} \rrbracket$	$\langle\!\langle 1, \boldsymbol{\Omega} \rangle\!\rangle$	$(\!(\boldsymbol{\Omega}, \boldsymbol{\Omega})\!)$

TABLE 2.5. Fundamental Sign Laws of the Three Representations of Directed Quantities

Sign Law	Representation		
	Native	Correlated	Extremum
Migratory	$-\llbracket s, g \rrbracket = \llbracket -s, g \rrbracket$	$-\langle\!\langle s, g \rangle\!\rangle = \langle\!\langle -s, -g \rangle\!\rangle$	$-(\!(s, g)\!) = (\!(s, -g)\!)$
Unbinding	$\llbracket s, g \rrbracket = \llbracket s, -g \rrbracket$	—	$(\!(s, g)\!) = (\!(-s, -g)\!)$

Remark 2.18. For any $\mathbf{x} \in V^\bullet$ the ray $[\mathbf{x}]$ may be identified with one point of the projective geometry $\mathcal{P}(V)$; while the semiray $\llbracket \mathbf{x} \rrbracket$ may be identified with one of two opposite points of the oriented projective geometry $\mathcal{OP}(V)$, the other being $\llbracket -\mathbf{x} \rrbracket$.

2.6. **Equivalence Rules of the Three Brackets.** These notions are most transparent when the gsigns and gauges of brackets are restricted to the standard basis blades (multivectors) or their negatives. We call such blades *basic blades* and the brackets formed from them *basic brackets*. However, unless otherwise stated the results given below apply to brackets with general gsigns and gauges.

Table 2.5 gives the fundamental equivalence rules for the brackets of the three representations under certain sign changes. These are the *fundamental sign laws*. They divide into two types, the *migratory* and the *unbinding* sign laws.

The migratory laws govern how a negative sign outside a bracket moves inside and distributes over the gsign and gauge. The migratory sign laws for the native and extremum brackets are *semidistributive*. I use this term to mean that the external sign distributes to only one of the two parts: the gsign for the native bracket and the gauge for the extremum bracket. The migratory sign law for the correlated bracket is (fully) *distributive*: the external sign distributes to both the gsign and gauge.

The unbinding sign laws describe valid sign changes within a bracket that reverse the orientation of the bracket itself. For valid basic brackets with gsign s and gauge g the *bracket orientation* is given by $s \circledcirc g \in \{\pm 1, \pm \mathbf{\Omega}\}$. Correlated brackets representing odd quantities are fixedly bound to either $\mathbf{\Omega}$ or $-\mathbf{\Omega}$ during a calculation. Correlated brackets representing even quantities are always bound to $+1$ by definition. Therefore, correlated brackets have no unbinding sign law. The migratory sign law of the native bracket is also unbinding, although it is listed in only the migratory row of Table 2.5.

Now we discuss the sign laws that can be derived from the fundamental ones. We consider the simpler cases of the native and extremum brackets first. Altogether there are eight possible sign patterns for any bracket. By starting with the two fundamental equations involving four of these sign patterns, we can derive altogether two strings of equalities among four sign patterns. Therefore, for each gsign-gauge pair that define a valid odd or even bracket, the native and extremum brackets divide into two equivalence classes, each the negative of the other.

For native brackets we immediately derive the following sign equivalence laws (including the fundamental ones) from Table 2.5:

$$\begin{aligned} \llbracket s, g \rrbracket = \llbracket s, -g \rrbracket = -\llbracket -s, g \rrbracket = -\llbracket -s, -g \rrbracket \quad \text{and} \\ -\llbracket s, g \rrbracket = \llbracket -s, g \rrbracket = \llbracket -s, -g \rrbracket = -\llbracket s, -g \rrbracket. \end{aligned} \tag{2.10}$$

Equations (2.10) can be expressed more compactly in terms of the "plus-or-minus" $\pm$ and "minus-or-plus" $\mp$ symbols consisting of pairs of coordinated signs:

$$\begin{aligned} \pm \llbracket \pm s, g \rrbracket = \pm \llbracket \pm s, -g \rrbracket \quad \text{and} \\ \mp \llbracket \pm s, g \rrbracket = \mp \llbracket \pm s, -g \rrbracket. \end{aligned} \tag{2.11}$$

From these equations we see that native brackets with the same contents disregarding signs and the same sign parity disregarding the sign of the gauge are equal.

For extremum brackets we immediately derive the following sign equivalence laws (including the fundamental ones) from Table 2.5:

$$\begin{aligned} \llparenthesis s, g \rrparenthesis = \llparenthesis -s, -g \rrparenthesis = -\llparenthesis -s, g \rrparenthesis = -\llparenthesis s, -g \rrparenthesis \quad \text{and} \\ -\llparenthesis s, g \rrparenthesis = \llparenthesis s, -g \rrparenthesis = \llparenthesis -s, g \rrparenthesis = -\llparenthesis -s, -g \rrparenthesis. \end{aligned} \tag{2.12}$$

As in equations (2.11), these relationships can be expressed more compactly in terms of coordinated sign pair symbols:

$$\begin{aligned} \llparenthesis \pm s, \pm g \rrparenthesis = -\llparenthesis \pm s, \mp g \rrparenthesis \quad \text{and} \\ -\llparenthesis \pm s, \pm g \rrparenthesis = \quad \llparenthesis \pm s, \mp g \rrparenthesis. \end{aligned} \tag{2.13}$$

From these equations we see that the extremum brackets with the same contents disregarding signs and the same sign parity are equal.

Finally, we discuss correlated brackets. Because they are bound to an arbitrarily chosen but fixed orientation, $\mathbf{\Omega}$ or $-\mathbf{\Omega}$, and because the directions of the gsign and gauge of an even correlated bracket must be equal by definition, it is possible for a gsign-gauge pair that would be valid in a native or extremum bracket to be invalid in a correlated bracket. Therefore, the eight possible sign patterns produced by distributing plus and minus signs in the three positions, bracket, gsign, and gauge, sort into two pairs of two equations. One set of pairs contains four brackets that are all valid and the other contains four brackets that are all invalid under a given assignment of grade conforming gsigns and gauges. The valid pair of equations is further divided into two equations with the brackets in one of the equations being the negatives of the brackets in the other equation.

Thus we have the following set of equations with one or the other of the pairs of equations (2.14a) or (2.14b) valid depending on the values of the gsign s, the gauge g, and the binding orientation, $\mathbf{\Omega}$:

$$\begin{aligned} \langle\!\langle s, g\rangle\!\rangle &= -\langle\!\langle -s, -g\rangle\!\rangle \quad \text{and} \\ -\langle\!\langle s, g\rangle\!\rangle &= \langle\!\langle -s, -g\rangle\!\rangle, \end{aligned} \tag{2.14a}$$

or

$$\begin{aligned} \langle\!\langle s, -g\rangle\!\rangle &= -\langle\!\langle -s, g\rangle\!\rangle \quad \text{and} \\ -\langle\!\langle s, -g\rangle\!\rangle &= \langle\!\langle -s, g\rangle\!\rangle. \end{aligned} \tag{2.14b}$$

For example, if we let $s = \mathbf{e}_1$, $g = \mathbf{e}_{23}$, and bind the correlated brackets to $\mathbf{\Omega} = \mathbf{e}_{123}$, the following equation for an odd quantity is valid $\langle\!\langle s, g\rangle\!\rangle = -\langle\!\langle -s, -g\rangle\!\rangle$, but the equation $-\langle\!\langle s, -g\rangle\!\rangle = \langle\!\langle -s, g\rangle\!\rangle$ is invalid. Similarly if we let $s = -\mathbf{e}_1$ and $g = \mathbf{e}_1$, the following equation for an even quantity is valid $-\langle\!\langle s, -g\rangle\!\rangle = \langle\!\langle -s, g\rangle\!\rangle$, but the equation $\langle\!\langle s, g\rangle\!\rangle = -\langle\!\langle -s, -g\rangle\!\rangle$ is invalid.

We can summarize equations (2.14) if we first make two definitions. Let the *external sign parity* of a correlated bracket be odd if it has an odd number of negative signs in front of it and even if it has an even number of negative signs in front of it. Analogously, let the *internal sign parity* of a correlated bracket be odd if it has an odd number of negative sign inside it and even if it has an even number of negative signs inside it. For example, the correlated bracket expressions $-\langle\!\langle -s, -g\rangle\!\rangle$ and $-\langle\!\langle s, g\rangle\!\rangle$ both have odd external sign parity and even internal sign parity. Then from these equations we see that correlated brackets with the same contents disregarding signs, and with opposite external sign parity but equal internal sign parity are either equal or both invalid.

Equations (2.14) are written in parallel fashion with respect to the unbinding sign law for native brackets $[\![s, g]\!] = [\![s, -g]\!]$. Thus, if we had native instead of correlated brackets, the upper equations of equation pair (2.14a) and equation pair (2.14b) would be equated, similarly the bottom equations in these two equation pairs would also be equated. Thus, neither equation pair (2.14a) nor equation pair (2.14b) would be invalid.

2.7. **William's Twisted Notation.** So close. A brilliant innovator.

Another clue. Lounesto et al. [119] introduced the concept of directed quantity. Line of action. Cite Jancewicz.

2.8. **Some Tensor Theoretical Stuff.** In this paper we follow the approach of Åberg in his research report [1, p. 30]. Thus we view both the Clifford algebra and the OC algebra as the exterior algebra with different multiplications. Therefore we begin with a thorough foundation for the exterior algebra.

Using the following passage paraphrased from (and further details from) Trautman's talk notes [178, p. 3] the approach of Yokonuma's book [203] to tensor densities, and pseudo-tensors can be related to sections of bundles.

> Recall two ways of describing vector fields on a m-dimensional manifold: 1. Start from principal bundle P of linear frames. Vector field X is an equivariant map. 2. Tangent bundle: Vector field X' is a section of the tangent bundle. Connection between the two: 1⇒2

by forming associated bundle; 1⇐2 by defining a linear isomorphism.

$$\boxed{\text{Group}} \to \boxed{\text{Ring}} \to \boxed{\text{Field}} \to \boxed{\text{Vector Space}} \to \boxed{\text{Algebra}}$$

For a full mathematical treatment of the exterior algebra as the quotient of a tensor algebra by an ideal see the book by Dummit and Foote [64].

The following definition of an *R-algebra* is taken directly from the book by Dummit and Foote [64, p. 323].

Definition 2.19. Let R be a commutative ring with identity. An *R-algebra* is a ring A with identity together with a ring homomorphism $f\colon R \to A$ mapping 1_R to 1_A such that the subring $f(R)$ of A is contained in the center of A.

Lemma 2.20 (3.1.6 verbatim from Rossmann's book [151, pp. 134 f.])**.** *There is a one-to-one correspondence between differential k-forms and alternating $(0, k)$-tensors so that the form $f_{ij\ldots}dx^i \wedge dx^j \wedge \cdots$ $(i < j < \cdots)$ corresponds to the tensor $T_{ij\ldots}dx^i \otimes dx^j \otimes \cdots$ defined by*

(1) *$T_{ij\ldots} = f_{ij\ldots}$ if $i < j < \cdots$ and*
(2) *$T_{ij\ldots}$ changes sign when two adjacent indices are interchanged.*

Proof. As noted above every k-form can be written *uniquely* as

$$f_{ij\ldots}dx^i \wedge dx^j \wedge \cdots \ (i < j < \cdots)$$

where the sum goes over *ordered* k-tuples $i < j < \cdots$. It is also clear that an alternating $(0, k)$-tensor $T_{ij\ldots}dx^i \otimes dx^j \otimes \cdots$ is *uniquely* determined by its components

$$T_{ij\ldots} = f_{ij\ldots} \text{ if } i < j < \cdots$$

indexed by ordered k-tuples $i < j < \cdots$. Hence the formula

$$T_{ij\ldots} = f_{ij\ldots} \text{ if } i < j < \cdots$$

does give a one-to-one correspondence. The fact that the $T_{ij\ldots}$ defined in this way transform like a $(0, k)$-tensor follows from the transformation law of the dx^i. □

Clifford Opposite Algebra. See Deligne's *Notes on Spinors* [55, p. 107 (phys. p. 9)] for "opposite" in ungraded vs. super sense, and [55, p. 108 (phys. p. 10)] "**(E)** The identity of V extends to isomorphism from the opposite of the super algebra $C(Q)$ to the super algebra $C(-Q)$."

Levi-Civita Tensor $\epsilon_{i_1\cdots i_n}$ vs. the Tensor Density $\varepsilon_{i_1\cdots i_n}$. See Pope's *Geometry and Group Theory* [142, pp. 55 (phys. p. 56)].

Clifford Algebra Isomorphic to Exterior Algebra. For this and more in the context of quantum theory and modern mathematical physics (similar to Åberg, Crumeyrolle, Fauser, or Oziewicz) see Roepstorff and Vehns' *An Introduction to Clifford Supermodules* [150, pp. 6 ff.]. Look especially at the source [150, pp. 6] of this quote "...or phrased differently, that the above CAR algebra is irreducibly represented on the superspace $\bigwedge V$."

Some bookmarks from Melrose's lecture notes of the Graduate Analysis Seminar, 18.199 (Spring 2006) [131] follow:

(1) Ricardo Andrade, Clifford algebras, Pin and Spin groups, pp. 3 f.;

(2) Yakov Shapiro, Clifford modules and connections, pp. 4 f.;
(3) Ricardo Andrade, Periodicity, p. 9;
(4) "Theorem 3. For an oriented manifold manifold M the SO bundle given by oriented orthonormal frames has a spin structure,..." p. 11;
(5) Yakov Shapiro, Z2-grading, p. 12.

Here is a quote from Chapter 3, *Coordinate Invariance and Manifolds*, of Melrose's lecture notes for his class Graduate Analysis, 18.156 (Spring 2007) [132, p. 5, "35"]:

> **2. Manifolds.**
> I will only give a rather cursory discussion of manifolds here.... There are in fact several different, but equivalent, definitions of a manifold.
> **2.1. Coordinate covers.** ...
> **2.2. Smooth functions.** ...
> **2.3. Embedding.** ...

$$A \ulcorner B \tag{2.15}$$

$A \ulcorner B$
$T^{A \ulcorner B}$
$T^{X^{A \ulcorner B}}$

$$A \neg B \tag{2.16}$$

$A \neg B$
$T^{A \neg B}$
$T^{X^{A \neg B}}$

$$A \llcorner B \tag{2.17}$$

$A \llcorner B$
$T^{A \llcorner B}$
$T^{X^{A \llcorner B}}$

$$A \lrcorner B \tag{2.18}$$

$A \lrcorner B$
$T^{A \lrcorner B}$
$T^{X^{A \lrcorner B}}$

One theoretical basis for these reduction rules is given by the more complicated [9] formulation of multilinear algebra, tensor algebra, and the quotients of tensor algebras by ideals. This is an important approach, but [10] too elaborate for this introductory paper. We can give a relatively simple, generators and relations axiomatic basis for all these algebras as algebras of a quadratic form (or its associated nondegenerate [11] symmetric bilinear form). From such axioms we can derive as theorems various reduction rules important for calculation. Instead we will use a

---

[9] $A \lrcorner B$
[10] $T^{A \lrcorner B}$
[11] $A \llcorner B$

hybrid of reduction rules and the Clifford-like characterization of the OC [12] algebra with occasional reference to the Chevalley isomorphism of the linear spaces of the exterior, Clifford, and orientation congruent algebras (known among some theoretical physicists under the name the Kähler-Atiyah algebra according to Crumeyrolle in his book [51, p. 44]. We bring in the Clifford-like formulation of the OC algebra. Formulation by the Chevalley isomorphism also gives a clean, not too complicated, theoretical formulation for the exterior and Clifford algebras, but we still cannot include the OC algebra because of its nonassociativity. (Maybe false?) Clearly the Hopf algebraic approach encompasses all three algebras. While it uses an intriguing theory that allows a nonsymmetric (degenerate, too?) bilinear form and also leads to important algorithms for computing the products of these algebras, it is still to complicated for this paper.

the exterior products $\mathbf{v} \wedge \mathbf{w}$ of vectors $\mathbf{v}$, $\mathbf{w}$ in some vector space V of finite dimension n. The set of all such products, their scalar multiples, and their sums is another vector space of dimension 2^n, symbolized as $\bigwedge V$, and the associative algebra naturally defined on it is called the *exterior algebra* also written with the same symbol.

$$\bigwedge V + \bigwedge V \tag{2.19}$$

The wedge product pp. 211 ff. (223 ff.) of smooth.djvu Intro to Smooth Manifolds, John M. Lee.

Mathematics for Physics I, Michael Stone, amaster.pdf, App. A, Linear Algebra Review, pp. 396 ff. (406 ff.)

---

[12] $T^{A \llcorner B}$

3. Oriented Differential Forms

> given an n-dimensional vector space E, whether or not it has an inner product, one can always construct the dual vector space E^*, and the construction has nothing to do with a basis in E.
>
> Theodore Frankel [76, p. 44]

> Whenever we consider or deal with some linear space L, over say field $\mathbb{R}$ or $\mathbb{C}$, ... then we have on hand automatically, independent of our wishes, another linear space $L^* = \mathrm{Hom}(L, \mathbb{R})$, dual to L, of all linear mappings
>
> $$L^* \ni \alpha\colon L \ni v \to \alpha v \in \mathbb{R}. \ldots$$
>
> To say that we do not need to consider "any" dual space is meaningless because, independently of our wishes, dealing with L we have to do at least with the pair $\{L, L^*\}$.
>
> Zbigniew Oziewicz [135, pp. 245 f.]

Before we can understand why we need odd differential forms and how to calculate with them we must learn their names, appearances, and origins. We begin with an intuitive, geometry-based approach.

The concept of an odd differential form may be traced to an analogous tensorial version given by Weyl in his book [198]. Although, the related concept of *outer orientation* appears at least as early as Veblen's pioneering topology book [190, pp. 10, 194], and Veblen and Whitehead's differential geometry book [191, pp. 55 f.]. An *outer orientation* can be given a neat, modern definition in terms of quotient spaces and the ordinary, inner orientation of a vector space. This is described by Shaw in his book [166, p. 78].

Burke, first, in the paper [33], later, in the book [34], and finally, in the two draft papers [35] and [36], became the strongest, most recent advocate for the formulation of electrodynamics using both even (ordinary) and odd (twisted) differential forms. Unfortunately, William Lionel Burke died at age 55 in 1996 from a cervical fracture that he suffered in an automobile accident. See his Wikipedia entry [199] for more information.

3.1. **Imaging Differential Forms.**

We begin with some facts about the existence and nature of the dual space of covectors from which differential forms are constructed. These facts are emphatically stated in a couple of quotations. First, we quote Frankel from *The Geometry of Physics* [76, p. 44]:

> ... given an n-dimensional vector space E, whether or not it has an inner product, one can always construct the dual vector space E^*, and the construction has nothing to do with a basis in E.

In this line Oziewicz says in Reference [135, pp. 245 f.]:

> Whenever we consider or deal with some linear space L, over say field $\mathbb{R}$ or $\mathbb{C}$, ... then we have on hand automatically, independent of our wishes, another linear space $L^* = \mathrm{Hom}(L, \mathbb{R})$, dual to L, of all linear mappings
>
> $$L^* \ni \alpha\colon L \ni v \to \alpha v \in \mathbb{R}.$$

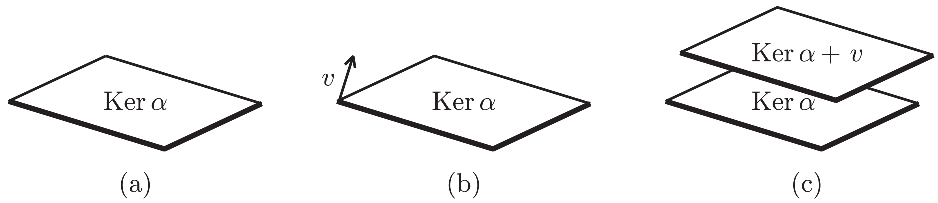


FIGURE 3.1. Constructing a Picture of a Covector

> ... To say that we do not need to consider "any" dual space is meaningless because, independently of our wishes, dealing with L we have to do at least with the pair $\{L, L^*\}$. In fact all tensor products are involved.

Having been assured that the dual space (along with its antisymmetric tensor products that make up differential forms) exists and cannot be ignored, we inquire about picturing exterior differential forms. On the contrary, Yang uses his review [202, pp. 968 f.] of the book *Applied Differential Geometry* [34] by Burke to unequivocally shun the practice of drawing pictures of differential forms. We do not follow his advice, but we must realize the pictures we draw are somewhat fraudulent.

A degree of fraud enters because there is no way to draw a p-form in the same picture with a q-vector. Localized to a single point of a manifold, a differential p-form becomes a p-covector, a member of the space of the exterior algebra of the cotangent space. This space is dual to space of the exterior algebra of the tangent space containing q-vectors. Although, these spaces are linked by duality, they are separate. Yet, in their publications Misner, Thorne, and Wheeler [133], Burke [33, 34], and Schouten [160, 162], to name a few, all seem to be drawing exterior differential forms. So what are these authors really picturing?

For an answer we turn again to Oziewicz, who continues in his paper [135, p. 246]:

> ... it is important to understand the geometrical representation of the covector $\alpha \in L^*$ as the codimension one hyperplane $\operatorname{Ker}\alpha \equiv \{\, v \in L, \alpha v = 0 \,\} \subset L$. In fact the covector α is completely and uniquely determined by $\operatorname{Ker}\alpha$ and any vector $v \in L$ such that $\alpha v = 1$. The wave fronts for instance are described by covectors (forms).

If we can draw the direct space of vectors only, our pictures of covectors are actually a representation of them in terms of vectors. From Oziewicz's recipe we construct these pictures in a 3-dimensional space as follows. First, as in Figure 3.1(a), for a given covector α we draw the hyperplane determined by the vectors in the kernel of α. Any vector wholly contained in this hyperplane is in $\operatorname{Ker}\alpha$. Next, as in Figure 3.1(b), we find some vector v that when evaluated with α gives unity $\alpha v = 1$. As Oziewicz says these two ingredients are all we need. However, for the usual picture of a 1-form as parallel "slicers" or "chopping blades," we translate a copy of the kernel hyperplane to a location at the tip of v. This gives the hyperplane $\operatorname{Ker}\alpha + v$. Finally, removing v completes the familiar drawing of a covector shown in Figure 3.1(c).

3.2. **What Are Odd Forms?** The prototypical odd form is the 3-form of unit volume density $\widetilde{\Omega}$. Here we have used the tilde, as we do throughout this paper, to indicate an odd p-form or p-vector. Suppose that we have some signed additive quantity such as electric charge that is distributed throughout a region R of ordinary physical space. Then this distributed charge can be represented as the charge density 3-form $Q\widetilde{\Omega}$. Evaluating this expression over a chain of composed of odd 3-vector or volume *capacities* gives the net (positively or negatively signed) charge Q_t contained within R.

We can derive all odd forms from the set of even forms and the *odd unit scalar* $\widetilde{1}$. Differential geometry provides us with a technique for representing the odd unit scalar $\widetilde{1}$ by pairing the unit scalar 1 with the top form as $(1, \Omega)$.

3.3. **The Many Names of Odd Forms.** As remarked in the Introduction differential, or linear, forms come in exactly two orientations. The well-known one is the *even* or *straight* orientation. The second, less commonly known one is the *odd* or *twisted* orientation. Several other terms are used in the literature to distinguish these two possible orientations of p-forms (and p-vectors), but in this paper we prefer to use any of the four words just mentioned.

The number of terms applied to differentiate the orientations of these objects may present some difficulty, which is only compounded when we encounter their tensor analytic representation later. Table 3.1 gathers some of the more common names for p-forms of both orientations together with some references in which they appear.

Names for forms are separated by their orientations into the first two columns of Table 3.1. To avoid further confusion we have simply labeled these "Column 1" and "Column 2." Each row presents pairs of complementary names that are used together to distinguish the two orientations. For emphasis the distinguishing terms are printed in boldface. Terms enclosed in parentheses tend to be optional. Authors sometimes substitute a generic word such as *ordinary* for the distinguishing terms of Column 1. Occasionally they write in "diagonal" nomenclature by mixing the Column 1 terms of some row in Table 3.1 with the Column 2 terms of another row.

The French terms in the top row of Table 3.1 seem to have been used first by de Rham. Although the specific reference cited here, his monograph *Variétés Différentiables* [53], may not be the source of their first appearance. The French is used too by some authors writing in English. In the second row we find the translations of de Rham's original French terms as they appeared in the English version of de Rham's book [54]. In the English translation we also find the slight variation of appending the distinguishing phrase *of even* (*odd*) *type* to the description of a p-form.

3.4. **The Schouten Icons.** Energy integral motivation.

Tensorial terms and notation. Densities

3.5. **Probing Electromagnetic Fields for the Native Geometric Structure.** The Maxwellian double plates.

The quantities of electrodynamics (and many, if not all, classical field theories) are naturally grouped as pairs with complementary physical and geometric properties. Let us express the field quantities in each of these pairs as differential forms. Then, physically, the exterior product of the forms in each pair always has the same

Table 3.1. Mathematical Terms for Differential Forms with the Two Orientations

Column 1	Column 2	References
pair p-form	***impair*** p-form	de Rham [53, pp. 19, 22 ff.]
even p-form	**odd** p-form	de Rham [54, pp. 17, 19 ff.], Jancewicz [108]
(**ordinary**) p-form	**twisted** p-form	Frankel [75, pp. XXf], Burke [33] & [34, pp. 151 ff., 183 ff.], Bossavit [21, pp. 67 ff.]
(**true**) p-form	**pseudo**-p-form	Jancewicz [105, 106], Frankel [76, pp. 86 f.]
polar p-form	**axial** p-form	Sorkin [170]

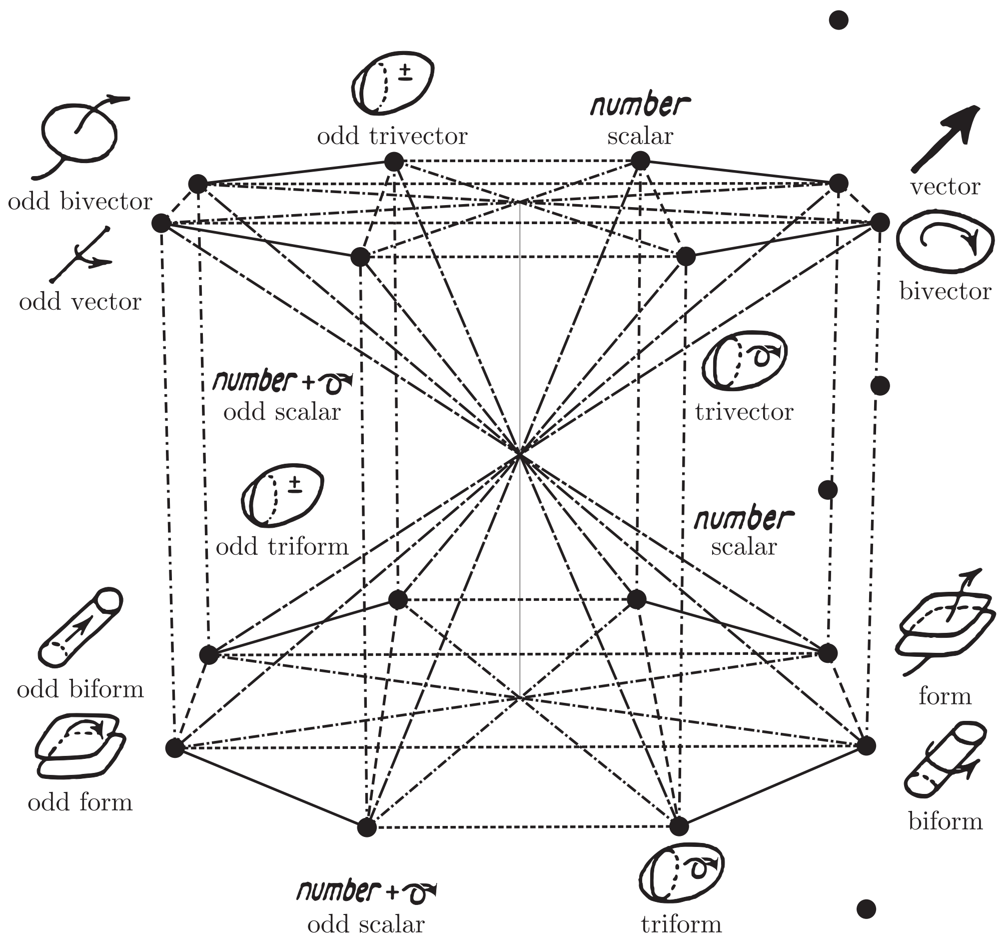


Figure 3.2. Schouten's Diagram Revised

dimension (measurable physical property). In electrodynamics this dimension is energy, although in other field theories it may be something else.

Geometrically, the degree of each product is the maximum or *top* degree, i.e. it is equal to the dimension of the linear space or manifold on which these differential forms are defined. Therefore, on an n-dimensional manifold, if one of the two forms in a pair is a p-form, the other is a $(n-p)$-form.

Furthermore, this geometric duality extends to the *orientations* of the forms. This fact follows quite naturally when we consider that, in three dimensional space, energy and volume are taken to be nonnegative quantities. Then, if space itself has no intrinsic orientation, the exterior product of two forms in a pair must possesses an *odd* or *twisted* orientation. Therefore, the orientations of the differential forms in these pairs must be complementary—one even, the other odd—to yield a product which is an odd form.

The physical and geometric duality we have just described is fundamental in Tonti's analysis of the mathematical structure of physical theories [175, 176, 177]. As it happens in his analysis, the field quantities represented by an *even* differential forms are the *configuration* variables and those represented by an *odd* form are the *source* variables. Tonti's scheme also embraces theories other than field theories, where, of course, the characterization of the physical quantities by the orientation of the differential forms representing them does not directly apply. Also, as Frankel [76, p. 122] points out, if the hypothetical magnetic monopole carrying magnetic charge is found, the correspondence we have given between even vs. odd orientations and configuration vs. source variables would be reversed. Nevertheless, even though we could use other corresponding pairs of terms such as *intensity* and *quantity* as given by Frankel [76, p. 122], or *intensive* and *extensive* variables, as inspired by thermodynamics, in this paper we adopt Tonti's terms configuration and source variable as generic labels for such physical quantities.

Certain classes of two-typed mathematical concepts may represent this dichotomy of physical quantities. Unfortunately, due to different theoretical underpinnings or just different conventions, a large number of pairs of terms have appeared in the literature.

Table 3.2. Some Mathematical Terms for Differential Forms Corresponding to Configuration and Source Physical Variables

Differential Forms		
Configuration Variable	Source Variable	References
(**straight**) p-form	**twisted** $(n-p)$-form	Burke [33, 34] Frankel [75]
pair p-form	***impair*** $(n-p)$-form	de Rham [53]
even p-form	**odd** $(n-p)$-form	de Rham [54] Jancewicz [108]
polar p-form	**axial** $(n-p)$-form	Sorkin [170]
(**true**) p-form	**pseudo**-$(n-p)$-form	Frankel [76] Jancewicz [105, 106]

Tables 3.2 and 3.3 present some of these many names for the mathematical objects corresponding to force and source variables together with some references. In these tables the distinguishing terms are in boldface. For some rows of Tables 3.2 and 3.3 the mathematical type representing a force variable is frequently distinguished by the absence of a term. Although, for these same rows, sometimes an

Table 3.3. Some Mathematical Terms for Tensors Corresponding to Configuration and Source Physical Variables

Tensors

Configuration Variable	Source Variable	References
covariant p-vector	contravariant **W**-p-vector **density** of weight $+1$	Schouten & van Dantzig [164], Post [143, pp. 503 f.], Tonti [175, p. 122]
contravariant (**ordinary**) $(n-p)$-vector **density** of weight $+1$	covariant **W**-$(n-p)$-vector	
covariant p-vector	contravariant p-vector **density** of weight $+1$	Schouten [160, p. 28], Schouten [162, pp. 29 ff.]
contravariant $(n-p)$-vector **$\boldsymbol{\Delta}$-density** of weight $+1$	covariant **W**-$(n-p)$-vector	
covariant (**absolute**) p-vector	contravariant **even relative** p-vector of weight $+1$	adapted from Kuptsov [116]
contravariant **odd relative** $(n-p)$-vector of weight $+1$	[not named]	
covariant (**true**) p-vector	contravariant **densitized** **pseudo**-p-vector of weight $+1$	adapted from Baez [6]
contravariant **densitized** (**true**) $(n-p)$-vector of weight $+1$	covariant **pseudo**-$(n-p)$-vector	

optional term is inserted. This term may be either a row-specific one (shown in parentheses) or the general one *ordinary* (not shown). Note that de Rham's [53] original French terms *pair* and *impair* mean simply *even* and *odd* in English [54].

For compactness in Table 3.2, the verbal variations which accompany the different degrees or orders of a differential form, such as *scalar*, *vector*, or *bivector*, are reduced to the general phrases p- or $(n-p)$-*form*. However, in the tensor analytic formulations of Table 3.3, the direct counterpart of a representative *even* p- or *odd* $(n-p)$-*form* becomes a *covariant* p- or $(n-p)$-*vector*, respectively, with a suitably descriptive phrase or symbol added. The generality of tensor analysis encourages us to consider another counterpart to both differential form representatives, thus doubling the entries in Table 3.3. These alternative species, dual to the direct tensorial counterparts, were described as *densitized* by Baez [6]. They appear in Table 3.3 as *contravariant* $(n-p)$- and p-*vectors* with ranks complementary to the direct tensors.

We should also mentioned here that the term *p-vector* always denotes a *totally antisymmetric* tensor, the only species of interest in this paper. In a more general context, a phrase such as *contravariant p-vector density of weight* $+1$ would be replaced with *contravariant tensor density of weight* $+1$ *and valence p*. Even more generally, a *true* or *absolute* tensor transforms as

(3.1)

and a *pseudo*- or *relative tensor* transforms as

$$\bar{p}^{\,i_1\cdots i_m}_{\,j_1\cdots j_n} = \tau(x)\, p^{\,l_1\cdots l_m}_{\,k_1\cdots k_n} \frac{\partial \bar{x}^{i_1}}{\partial x^{l_1}} \cdots \frac{\partial \bar{x}^{i_m}}{\partial x^{l_m}} \cdot \frac{\partial x^{k_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{k_n}}{\partial \bar{x}^{j_n}} \tag{3.2}$$

Because of its unique combination of simplicity and comprehensiveness, the best physical examples for introducing the machinery of exterior calculus come from classical electrodynamic theory. This is especially true of the modified theory of Maxwell's equations rewritten in symmetrical form to include the as yet unobserved magnetic monopole carrying a hypothetical magnetic charge and the field components due to it.

This is a study of the odd (also called twisted) differential forms generated by the classical field theories of physics. We consider the following questions:

- What are more natural geometric representations of these odd forms?
- What new algebraic formalisms reflect these geometric representations?
- How can exterior algebra and calculus be done in these new formalisms?
- What are some applications of this reformulated exterior calculus?

The mathematical basis of these theories has been under continuing development for over 250 years. Many workers have contributed to it from the early days of Euler to the present. For details of this history see [110, 111, 155].

We summarize the standard exterior calculus in the finite-dimensional case. First, we require an n-dimensional manifold M on which p-forms (antisymmetric tensor fields) ω_p exist. Also defined is the exterior derivative operator d which takes a p-form ω_p to a $(p+1)$-form ω_{p+1}. Introducing the exterior or wedge product $\wedge$, we may multiply a p-form ω_p and q-form ω_q to yield a $(p+q)$-form ω_{p+q}. The action of the d operator on the exterior product must be defined to follow a kind of Leibniz rule $d(\omega_p \wedge \omega_q) = d\omega_p \wedge \omega_q + (-1)^p \omega_p \wedge d\omega_q$. The interior product (or contraction operator) i_X, where X is a vector field, allows the reduction of a p-form to a $(p-1)$-form as $i_X\omega_p = \omega_{p-1}$. At this point the Lie derivative may also be defined.

If the manifold M is equipped with a metric g and an n-dimensional volume form Ω we may define the Hodge star operator $\star$ as a map from a p-form to a $(n-p)$-form $\star\omega_p$. Next, a scalar product for differential forms is formulated in terms of the $\star$ operator and Ω. With a scalar product in hand the coderivative operator $d^\star$ may be defined as the adjoint (or dual) of d. Finally, we form the Laplace-Beltrami operator $\Delta = d + d^\star$, the generalization of the Laplacian of vector calculus.

According to Hawkins [86], the basic formalization of the exterior calculus was first laid down by Élie-Joseph Cartan in a paper on Pfaff's problem [41], in which he defined the differential form and gave explicit rules for the exterior product and derivative of forms. Later workers added to the repertoire

According to Hawkins [86], the calculus of exterior differential forms remained unformalized before 1899. It was Élie-Joseph Cartan, in a paper on Pfaff's problem [41], who first defined the differential form and gave explicit rules for the exterior product and derivative of forms. However, the manipulation "the things which occur under integral signs," as Flanders called them [72, p. 1], was a not unpracticed skill before 1899. Asada's review [5] recounts that in the early days, even without a complete formal framework for the exterior calculus, "Poincaré developed the theory of integration of 2- and 3-forms together with considerations on orientation."

Although the pioneering Cartan himself applied this new tool to physics, perhaps the first mathematician to address physicists and engineers in his discussion of applications was Harley Flanders. The first edition of Flanders' text *Differential Forms with Applications to the Physical Science* was published in 1963 [72].

Of course, which properties are relevant will depend on the application at hand, for example in the electromagnetic theory, the duality between the electric and the magnetic fields is important. To find the optimal discretisation from an algebraic point of view is the problem of interest here. [linked as "Disc Diff Geom, de Beauce, Sen", \070225\0610065.pdf]

These assertions are supported by Flanders' book [72]. His first physical example, occurring as early as page 16, employs the components of the electric field E_i and the magnetic field H_i in a discussion of the Hodge star operator. This example is then soon expanded on page 44 to the first full section treating a physical application, titled "Maxwell's Field Equations."

The attractiveness of the exterior calculus is so overwhelming that a number of electrical engineers have adopted it and related mathematics. I list below selected publications in chronological order from this field.

1970: Balabasubramanian, Lynn, and Sen Gupta [7]
1981: Deshamps [58]
1984: Engl [66]; Kotiuga [114]
1986: Baldomir [8]
1991: Bossavit [23]
1993: Baldomir and Hammond [9]
1995: Warnick, Selfridge and Arnold [195]
1997: Baldomir and Hammond [10]; Warnick, Selfridge and Arnold [196]

Warnick, Selfridge and Arnold's boundary projection operator method [195] is the subject of our later analysis. These authors also report having taught electromagnetic field theory to engineers using differential forms [196].

References: Extensive bibliography on differential forms in electromagnetics as of 1997 [197].

Some Caveats. I present a theory of the CGB exterior calculus but without the algebraic rigor found, for example, in Bourbaki [29] or Shaw [167]. Also, I do not go into the further realms of differential geometry. This work treats only a simple finite n-dimensional real affine space E^n with a fixed metric, a pseudo-Euclidean space. But that should be sufficient for moving to later sophistications such as manifolds, fiber bundles, connections, or sheaves.

Remark 3.1. If we drop the metric and generalize it to a differentiable manifold our humble pseudo-Euclidean space becomes a setting for the topological development of electromagnetic theory. In this treatment Maxwell's equations are initially expressed sans metric by integrals of differential forms. Metric effects are introduced later through the Hodge star operator and the connection. Thus Maxwell's equations keep the same simple form throughout the theory. This is the approach of the pioneering work of F. Kottler [115], Élie-Joseph Cartan [42], and David van

Dantzig [182, 183, 184, 185, 186, 187, 188]. For the most thoroughgoing exposition of this kind, see the book *Foundations of Classical Electrodynamics* by Hehl and Obukhov [89], or the shorter papers [87, 88, 90].

3.6. **Background Reading.** To understand the physical motivation of this paper (and the application at its end) you should be familiar with the space-time split (3+1)-dimensional Maxwellian theory of electromagnetism. We use only basic electromagnetic theory, but as expressed in even (straight) and odd (twisted) differential forms. Sources that provide this background, but without distinguishing even from odd forms, are Baldomir and Hammer [10], Bamberg and Sternberg [11], Deshamps [58], and ADD Warner et al. REMARKING that they may sometimes refer to odd forms.

Similar references that do respect the difference between even and odd forms include Bossavit [23], [24, (suppl.)], Burke [33, 34], [35, 36, (suppl.)], Ingarden and Jamiołkowski [100], as well as Hehl and Obukhov's comprehensive book mentioned above [89]. Some related papers by Hehl et al. are [88, 87, 90].

As is done in this paper, the depiction of geometric quantities is emphasized by Schouten in his book *Tensor Analysis for Physicists* [159]. Although he uses tensor terminology and notation there rather than exterior products or differential forms, Schouten provides archetypal illustrations of all the three-dimensional geometric quantities we discuss: the even and odd kinds of scalars, vectors, bivectors, and trivectors (these last three collectively termed *p-vectors*), and their dual counterparts. Burke's [33, 34] wonderful drawings are particularly inspiring. See Salgado [154] for a quick visual reprise of the last two authors' work. Jancewicz' paper [105] is notable for illustrating the exterior products between virtually all of the possible nonzero combinations of scalars and p-vectors in three dimensions, both even and odd. He also depicts some sums of even and odd p-vectors and the exterior products between many combinations of even and odd p-forms. In addition Jancewicz gives examples of these geometric quantities from physics, including electrodynamics. He recaps that paper in his next one [106], without discussing sums, but adds an application to electrodynamics.[13]

A primary reference on geometric algebra (Clifford algebra interpreted geometrically) is the rather abstract treatment of Hestenes and Sobczyk in their book *Clifford Algebra to Geometric Calculus* [97]. I found it useful to supplement that work with Harke's *An Introduction to the Mathematics of the Space-Time Algebra* [84], taken from his thesis. Hestenes' approach is more leisurely in the book *New Foundations for Classical Mechanics* [93]. Conradt has authored a quick on-line tour of geometric algebra [47].

3.7. **The Rest of the Intro.** However, the initial, humble birth of this theory does not preclude developments such as its use to resolve the controversy over the derivation of the parity of (3+1)-dimensional electromagnetic quantities from their 4-dimensional spacetime expressions.

Boolean ring of sets

Let us call these objects odd vectors and forms.

Lounesto et al. [119] introduced the concept of directed quantity.

[13] Also, in the paper [108] Jancewicz gives more electrodynamic applications; while in another paper [107] he extends his geometric analysis of electrodynamics to four-dimensional spacetime.

More insight into the boundary projection operator is in Karl Warnick's Ph.D. dissertation [193].

There is an analogy to rational numbers and fraction bar notation. We write an even or odd decomposable form as a pair of forms enclosed in a bracket denoting equivalence classes of those pairs. This *correlated grade bracket* (*CGB*), splits into a *geometric sign* (*GS*) form and an *oriented measure* (*OM*) form. The GS form itself represents an equivalence class of ray and vector subspaces associated with semi-oriented projective spaces. The OM form is just an ordinary form. The exterior product of CGBs resolves into the *GS product* for the GS parts and the ordinary exterior product for the OM parts. For the ray and vector subspaces corresponding to the GS form the GS product corresponds to their oriented symmetric difference. A nonassociative Clifford-like algebra, the *orientation congruent* ($\mathcal{OC}$) algebra, is modified to be the nonlinear, but associative, GS product.

This should be a citation with extra text. Let's cite something now. See [195, Extra text]. Citing Bouma [26].

Warnick, Selfridge, and Arnold [195] expediently modify a rule $n \wedge \{(\alpha_s, \Omega_s)\} = \{(\alpha, \Omega)\}$ that Burke ([34], pp. 192 f.) gives as required for pullback to commute with the exterior differentiation. The version they use $\{(\alpha_s, \Omega_s)\} \wedge n = \{(\alpha, \Omega)\}$ does not commute with d but does allow them to write the boundary conditions consistently for both twisted one- and two-forms ($\widetilde{D}$ and $\widetilde{H}$ in their application to electromagnetism).

It is a wonderful fact that the *geometric sign* (or *gsign* for short) product which we write as a diamond $\diamond$ can be realized by the product of an algebra related to Clifford algebra.

4. Discovering the Orientation Congruent Algebra

> The White Rabbit put on his spectacles. "Where shall I begin, please your Majesty?" he asked.
>
> "Begin at the beginning," the King said, gravely, "and go on till you come to the end: then stop."
>
> Lewis Carroll [40, p. 182]

This section is a gentle, and thus, slow, discovery-based introduction to the orientation congruent algebra roughly recapitulating my own path of successive (or simultaneous) generalizations and refinements. I thereby attempt to ease the transition of the reader's thinking from the familiar to the unfamiliar. Unfortunately, precision and convenience necessitates the definition of a somewhat large number of new terms and conventions. A general outline of this section follows.

We begin here with the ordered pair notation $(d\alpha, \Omega)$ for odd forms that William L. Burke used in an early paper [33] and then in his textbook *Applied Differential Geometry* [34]. In this notation $d\alpha$ is an even form and Ω is a top-dimensional n-form. This ordered pair notation is the basis for our *extremum representation* of both odd and even forms. Our discussion of the extremum representation gives us the first, small taste of the orientation congruent algebra.

However, our explorations really become interesting when we consider Burke's description of a new notation for odd forms that he used in the later draft paper *Twisted Forms: Twisted Differential Forms as They Should Be* [36]. We focus on an intermediate expression that occurs near the end of his description. Unfortunately, on the way to his final result, Burke passes through this expression without realizing its significance. This expression is the basis for our *correlated representation* of both odd and even forms.

The key to the rest of this paper lies in the algebraic relationships among the exterior products of odd and even forms expressed in these two representations, the correlated and extremum. By making some reasonable assumptions about these algebraic relationships we intuit the laws of a new algebra. We dub this algebra the *orientation congruent* (or $\mathcal{OC}$) algebra. In Section 5 we draw on the laws we have found here to form a generators and relations axiom system for the $\mathcal{OC}$ algebra.

The three-dimensional space in which we do our initial explorations of the orientation congruent algebra is ideal for this purpose, not only because the resulting multiplication table is small enough to be comfortably displayed and comprehended on one page, and not only because human intuition is naturally most familiar with three dimensions, but also because some complications that arise in even-dimensional spaces are absent. These complications will be addressed in a later section.

Some Conventions. Because they are easier to visualize, throughout this section we prefer to work with odd and even multivectors rather than odd and even multiforms. We write the set of vectors in a general, ordered basis for V^n as $\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$. However, unless otherwise noted, throughout this section $n = 3$.

We also adopt the compact *multi-index notation*[14] for basis multivectors so that, for example, $\mathbf{e}_{12} := \mathbf{e}_1 \wedge \mathbf{e}_2$. In this example, we have left out the separating comma in the sequence $1, 2$, but this omission will not create ambiguous expressions as long as $n \leq 9$. It is also convenient to use upper case indices such as the I in $\mathbf{e}_I$ to symbolize a complete multi-index. Then we have, for example, $\mathbf{e}_I = \mathbf{e}_{12}$ if and only if $I = 12$. For more flexibility, we depart from the usual multi-index convention by allowing multi-indices that are sequences of integers not necessarily ordered from least to greatest.

In the bracket notations introduced below, we use a *bold* upper case omega $\mathbf{\Omega}$ for the basis top-dimensional 3-vector associated with some general, ordered basis for V^3, so that (except when discussing the subalgebra $\mathcal{OC}_n(\mathbf{\Omega})$) $\mathbf{\Omega} := \mathbf{e}_{123}$. We take an *underlined* bold upper case omega $\underline{\mathbf{\Omega}}$ to mean a generic top-dimensional 3-vector that does not necessarily have the same sense of orientation or weight as the basis 3-vector $\mathbf{\Omega}$. Thus $\underline{\mathbf{\Omega}} = c\,\mathbf{\Omega}$, for some $c \in \mathbb{R}^\bullet = \{x \mid x \in \mathbb{R} \text{ and } x \neq 0\}$ and we may write, for example, $\underline{\mathbf{\Omega}}_1 = c_1\mathbf{\Omega}$ and $\underline{\mathbf{\Omega}}_1 = c_2\mathbf{\Omega}$ for some, not necessarily distinct, $c_1, c_2 \in \mathbb{R}^\bullet$. Similarly, it is also convenient to employ an underlined bold numeral one $\underline{\mathbf{1}}$ as a generic symbol for a nonzero scalar. Thus, for example, $\underline{\mathbf{1}}_1 = c_1$ and $\underline{\mathbf{1}}_2 = c_2$ for some, not necessarily distinct, $c_1, c_2 \in \mathbb{R}^\bullet$.

4.1. **From Ordered Pairs to Extremum Brackets.** In *Twisted Forms* [36] Burke writes the ordered pair $(d\alpha, \Omega)$ representing an odd differential form with curly brackets as $(d\alpha, \{\Omega\})$ to indicate that $\{\Omega\}$ is an *orientation* as defined by the equivalence class, called a *ray*, under the reduction rules

$$\begin{aligned} \{d\alpha\} &= \{k\,d\alpha\} \text{ for all } k > 0, \text{ and} \\ -\{d\alpha\} &= \{-d\alpha\}. \end{aligned} \tag{4.1}$$

Burke uses the symbols $d\alpha$, $d\beta$, and so on, from the beginning of the Greek alphabet to indicate an exterior product of an indeterminate number of basis 1-forms. However, the $d\alpha$ here could actually be any simple (or decomposable) geometric object. So let us switch notations and carry on with multivectors rather than differential forms. The next two examples are taken from Burke's *Twisted Forms* [36], but translated into multivectors

Example 4.1. We can write the two orientations of two-dimension space V^2 as $\{\mathbf{e}_{12}\}$ and $-\{\mathbf{e}_{12}\} = \{-\mathbf{e}_{12}\} = \{\mathbf{e}_{21}\}$.

Example 4.2. An example of an odd vector in V^2 as written in this representation is $(\mathbf{e}_1, \{\mathbf{e}_{12}\}) = -(\mathbf{e}_1, \{\mathbf{e}_{21}\})$.

An orientation under the reduction rules of Equations (4.1) has the characteristics an equivalence class determined by the *signum function* sgn. The signum function $\operatorname{sgn}\colon \mathbb{R} \to \{-1, 0, 1\}$ is a surjection that maps its argument x to a number representing the *negative, neutral*, or *positive sign* of x according to

$$\operatorname{sgn} x = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

The signum function partitions its domain $\mathbb{R}$ into the preimages under sgn of each one-element subset of $\{-1, 0, 1\}$ (the fibers under sgn of $\{-1\}$, $\{0\}$, and $\{1\}$) which are the three familiar equivalence classes $\operatorname{sgn}^{-1}[\{-1\}]$, $\operatorname{sgn}^{-1}[\{0\}]$, and $\operatorname{sgn}^{-1}[\{1\}]$ of negative, zero, and positive real numbers. Similarly, the curly brackets partition the set of all simple multivectors into one special equivalence class containing the zero vector, the kernel of $\{\bullet\}$, and a set of two (for $n = 1$) or an infinite number (for $n \geq$

[14] See, for example, Shaw [167, pp. 326 f.].

2) of other infinitely-large equivalence classes which can be grouped (noncanonically, if $n \geq 2$) into two sets which are the "negatives" of each other.

Therefore, it is appropriate to write the orientation given by a top-dimensional n-form or n-vector in the first position just as the sign, positive or negative, of a real number proceeds it. Also, rather than indicating the ordered pair with a pair of enclosing parentheses, and indicating the equivalence relation with curly brackets, let us simply substitute a pair of enclosing double parentheses as in $(\!(\bullet, \bullet)\!)$. Combining these two conventions we have

$$(\!(\underline{\mathbf{\Omega}}, \mathbf{e}_I)\!) := (\mathbf{e}_I, \{\underline{\mathbf{\Omega}}\}), \tag{4.2a}$$

or, for a general multivector v,

$$(\!(\underline{\mathbf{\Omega}}, v)\!) := (v, \{\underline{\mathbf{\Omega}}\}). \tag{4.2b}$$

Example 4.3. In this double parentheses notation the odd vector of V^2 from Example 4.2 becomes $(\!(\mathbf{e}_{12}, \mathbf{e}_1)\!) = (\mathbf{e}_1, \{\mathbf{e}_{12}\})$.

Terminological Digression

We assign the representation of odd forms written in this double parentheses notation the full name, *unbound extremum bracket.* For convenience, we usually write simply *extremum bracket* instead of *unbound extremum bracket.*

This representation is *unbound* because we may use either $\mathbf{\Omega}$ or $-\mathbf{\Omega}$ (or any nonzero multiple of them) in the first position. Even though V^n may have an ordered basis which thus defines an orientation, an unbound representation is *not* restricted by this fact.

This representation is *extremum* because the first position contains an n-vector constructed from a n-dimensional base space. Such a multivector has the *maximum* possible degree (or *grade* in Clifford algebra jargon), namely, n. Later in this Subsection, when we allow the first position to also contain scalars, the element in that position may have the *minimum* possible grade, 0. In either case the first position of an extremum bracket will always contain an element with an *extremum* grade.

As is done with the terms *Peano bracket* and *Dirac bracket*, we use the singular, *bracket*, for each set consisting of an enclosing pair of doubled parentheses and their contents, and the plural, *brackets*, for more than one such set. We also employ *bracket* as a generic name for not only the extremum bracket, but also the correlated and native brackets introduced later. Each of these three are distinguished by their different enclosing, doubled brackets.

Rather than using the awkward phrases *the contents of the first (second) position*, let us adopt some short, meaningful, and—I hope—catchy names for them. Call the element in the first position of any (extremum, correlated, or native) bracket, which always has a role in determining an algebraic sign, the *generalized sign*, or simply *gsign.* Call the element in the second position of any bracket, which always has a role in determining a scalar magnitude, measure, or weight, the *gauge.*

End of Terminological Digression

We now consider the fundamental reduction rules for the extremum bracket. On page 189 of *Applied Differential Geometry* [34] Burke gives one such rule for odd forms. We rewrite it using an extremum bracket containing multivectors as

$$(4.3) \qquad \langle\!\langle \underline{\mathbf{\Omega}}, v \rangle\!\rangle = \langle\!\langle -\underline{\mathbf{\Omega}}, -v \rangle\!\rangle,$$

where v is a general multivector.

Equation (4.3) is example of an *unbinding* sign law. It is called this because it allows the gsign to invert. In the extremum bracket for an odd multivector the gsign is directly related to the n-vector $\mathbf{\Omega}$ which determines an orientation for the base space V^n.

We need one more fundamental reduction rule for extremum brackets. This next rule is called a *migratory* sign law because it describes what happens when a negative sign moves from outside to inside the brackets and vice versa, but without inverting the gsign:

$$(4.4) \qquad -\langle\!\langle \underline{\mathbf{\Omega}}, v \rangle\!\rangle = \langle\!\langle \underline{\mathbf{\Omega}}, -v \rangle\!\rangle,$$

where v is a general multivector.

We could now derive a complete set of reduction laws for the extremum bracket from these two fundamental ones. However, we will wait until later after all three brackets have been introduced, then derive a complete set of reduction laws for all three in succession. Next, we look at the rules for calculating the exterior products of odd and even multivectors.

On page 192 of *Applied Differential Geometry* [34] Burke gives a rule for the odd form that results from the exterior product of an odd form and even form. We rewrite it for multivectors in extremum brackets as

$$(4.5) \qquad \langle\!\langle \underline{\mathbf{\Omega}}, u \rangle\!\rangle \wedge v = u \wedge \langle\!\langle \underline{\mathbf{\Omega}}, v \rangle\!\rangle = \langle\!\langle \underline{\mathbf{\Omega}}, u \wedge v \rangle\!\rangle,$$

where u and v are general multivectors.

Curiously, I could not find an explicit expression for the exterior product of two odd forms in Burke's writings. Even Jancewicz, whose book chapter [105, pp. 408–410] and paper [106, pp. 252–255] treat such exterior (or *outer*) products in three dimensions with detailed graphical and symbolic examples, does not provide one. However, such an expression is also implicit in his writings. Bossavit, in his applied differential geometry compendium [24, p. 13], does give an explicit expression for the exterior product of two odd forms. He describes it in terms of the affine ratio of n-forms. The equivalent extremum bracket expression for the exterior product of two odd multivectors is

$$(4.6) \qquad \langle\!\langle \underline{\mathbf{\Omega}}_1, u \rangle\!\rangle \wedge \langle\!\langle \underline{\mathbf{\Omega}}_2, v \rangle\!\rangle = \operatorname{sgn}(r)\, u \wedge v \quad \text{for } r \in \mathbb{R} \text{ such that } \underline{\mathbf{\Omega}}_1 = r\, \underline{\mathbf{\Omega}}_2,$$

and where u and v are general even multivectors.

The last case is the familiar exterior product of two even multivectors which, following the conventions observed so far, could remain unchanged. However, we advance by representing a general even multivector such as u by a compatible extremum bracket which has a nonzero *scalar* rather than an n-vector gsign:

$$(4.7) \qquad u \cong \langle\!\langle \underline{\mathbf{1}}, u \rangle\!\rangle,$$

where the relation symbol $\cong$ is used to indicate representations that are isomorphic to each other.

Now the exterior product of general even multivectors u and v becomes

$$(4.8) \qquad \langle\!\langle \underline{\mathbf{1}}_1, u \rangle\!\rangle \wedge \langle\!\langle \underline{\mathbf{1}}_2, v \rangle\!\rangle = \langle\!\langle \underline{\mathbf{1}}_1 \cdot \underline{\mathbf{1}}_2, u \wedge v \rangle\!\rangle \cong \operatorname{sgn}(\underline{\mathbf{1}}_1 \cdot \underline{\mathbf{1}}_2)\, u \wedge v,$$

where the product denoted by a centered dot is ordinary real number multiplication.

Table 4.1. The Multiplication Table for the Subalgebra $\mathcal{OC}_n(\mathbf{\Omega})$ Generated by $\mathbf{\Omega}$ in the Orientation Congruent Algebra $\mathcal{OC}_n$ for Any Odd n.

		b	
	$a \circledcirc b$	1	$\mathbf{\Omega}$
a	1	1	$\mathbf{\Omega}$
	$\mathbf{\Omega}$	$\mathbf{\Omega}$	1

Before considering the remaining types of exterior products of odd and even multivectors in this light, we pause for a remark. The two fundamental reduction rules, the unbinding and migratory sign laws, given previously by Equations 4.3 and 4.4 for an odd multivector written as an extremum bracket with the maximum grade n-vector gsign $\underline{\mathbf{\Omega}}$ also apply to an even multivector written as an extremum bracket with the minimum grade nonzero scalar gsign $\underline{\mathbf{1}}$.

Using the extremum bracket for even multivectors, we can also easily rewrite Equation 4.5 for the odd multivector that results from the exterior product of a general odd and a general even multivector as:

$$(\!(\underline{\mathbf{\Omega}}_1, u)\!) \wedge (\!(\underline{\mathbf{1}}, v)\!) = (\!(\underline{\mathbf{1}}, u)\!) \wedge (\!(\underline{\mathbf{\Omega}}_1, v)\!) = (\!(\underline{\mathbf{\Omega}}_1 \cdot \underline{\mathbf{1}}, u \wedge v)\!) = (\!(\underline{\mathbf{\Omega}}_2, u \wedge v)\!), \tag{4.9}$$

where, again, the product denoted by a centered dot is ordinary real number multiplication, and u, v are general even multivectors.

It is more difficult to rewrite the product of two general odd multivectors given by Equation 4.6 so that the even multivector is represented by an extremum bracket. But this is so only because we wish to calculate the gsign of this product by multiplying the gsigns of the factors using some algebra. Such an algebra cannot have a product that is equivalent to the multiplication of real numbers, although the real numbers and their multiplication must be a subalgebra of it.

The necessary algebra turns out to be the *orientation congruent algebra*. However, at this point we do not need the $\mathcal{OC}$ algebra in its full generality. In fact, we can define it in Table 4.1 by the simple multiplication table of the subalgebra $\mathcal{OC}_n(\mathbf{\Omega})$ generated by $\mathbf{\Omega}$ in the orientation congruent algebra $\mathcal{OC}_n$ for any odd n. This subalgebra also has a simple analytic expression in terms of the Clifford algebra $\mathcal{C}\ell_n$:

$$u \circledcirc v = u \circ v^{\dagger} \quad \text{for all } u, v \in \mathcal{OC}_n(\mathbf{\Omega}). \tag{4.10}$$

Here we have followed Rota and Stein who, in their paper [153], use the small circle $\circ$ to represent the Clifford (or *circle*) product. We have also followed Hestenes and Sobczyk who, in their book [97, p. 5], use the dagger † to represent the *reversion* (or *main anti-automorphism* [51, p. 30]) operation of a Clifford algebra.

The subalgebra $\mathcal{OC}_n(\mathbf{\Omega})$ is isomorphic to $\mathcal{OC}_1$, one of the two smallest nontrivial orientation congruent algebras of a nondegenerate quadratic form. The other such orientation congruent algebra is $\mathcal{OC}_{-1}$ which, along with the Clifford algebra $\mathcal{C}\ell_{-1}$, is isomorphic to the complex numbers $\mathbb{C}$.

The subalgebra $\mathcal{OC}_n(\mathbf{\Omega})$ and the orientation congruent algebra $\mathcal{OC}_1$ are also both isomorphic to another algebra which may not be as familiar as the complex numbers, namely, the *double numbers*. The double numbers are known by many names in the literature. A very short introduction to them under the names *double-ring* and *Study numbers* is found in Lounesto's book [123, pp. 23 f.]. See also the *Wikipedia* article *Split-complex number* [201]. The relationship of the double numbers to exterior and other products involving odd multivectors and multiforms is treated in detail by Jancewicz's book chapter and paper [105, 106].

We can now express the exterior product of two general odd multivectors completely in extremum brackets:

$$(\!(\underline{\mathbf{\Omega}}_1, u)\!) \wedge (\!(\underline{\mathbf{\Omega}}_2, v)\!) = (\!(\underline{\mathbf{\Omega}}_1 \circledcirc \underline{\mathbf{\Omega}}_2, u \wedge v)\!) \cong \operatorname{sgn}(\underline{\mathbf{\Omega}}_1 \circledcirc \underline{\mathbf{\Omega}}_2)\, u \wedge v, \tag{4.11}$$

where u and v are general multivectors and the circled circle $\circledcirc$ is the orientation congruent product.

It is even more gratifying to see that using the orientation congruent algebra we can neatly condense all of Equations (4.8), (4.9), and (4.11), expressing various exterior products of general odd and even multivectors, into a single equation:

$$u \wedge v \cong (\!(s_u, g_u)\!) \wedge (\!(s_v, g_v)\!) = (\!(s_u \circledcirc s_v, g_u \wedge g_v)\!) \cong \operatorname{sgn}(s_u \circledcirc s_v)\, g_u \wedge g_v, \tag{4.12}$$

for general odd or even multivectors $u \cong (\!(s_u, g_u)\!)$ and $v \cong (\!(s_v, g_v)\!)$.

4.2. **From William's Twisted Notation to Correlated Brackets.** In *Twisted Forms* [36] Burke was attempting to overcome the limitations of the very successful $(d\alpha, \Omega)$ ordered pair notation for odd forms that he had used earlier by introducing, "what I have found to be the best and simplest notation for twisted forms." The defects he objects to are implicit in the following paragraph taken from *Applied Differential Geometry* [34, p. 184].

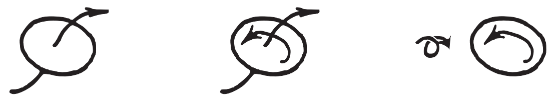
> I will discuss twisted tensors in two ways: first, in terms of their intrinsic properties, to emphasize that they are geometric objects and are as natural and fundamental as ordinary tensors; then in terms of a representation involving ordinary tensors, which is the easiest representation to manipulate although it unfortunately makes them seem like subsidiary objects. A careful definition of twisted tensors will appear only in the second discussion.

Here his "representation involving ordinary tensors" specialized to differential forms refers to the $(d\alpha, \Omega)$ notation for odd forms. The problem with it is twofold: This notation does not represent the "intrinsic properties" of odd forms; and, because it is a composite of even forms, it makes odd forms "seem like subsidiary objects."

Graphically, he has a fine intrinsic (or *native*) representation of odd tensors. I have traced it as far back as the original 1924 German language version of Schouten's *Ricci Calculus* [158, p. 22]. An example of an odd bivector in the native graphical representation appears in Figure 4.1(a). Burke exploits the native graphical representation fully in his later publications [33, 34, 35, 36] along with graphical representations of many other geometric quantities and their relationships. But analytically, symbolically, he must fall back on the $(d\alpha, \Omega)$ or similar notation.

4.3. **From Bracket Interconversions to the $\mathcal{OC}$ Algebra.** Need to say: add some specific examples from my old notes to motivate the idea of using a Clifford-like algebra because the Clifford product is similar to the symmetric difference operation on sets. But the sign needs to be modified.

Need to say: only scalars and blades make sense in correlated brackets.



(a) native representation (b) correlated representation (c) extremum representation

Figure 4.1. The Three Standard Graphical Representations of an Odd Bivector in V^3. Figure 4.1(a) was slightly modified from Schouten's book [162, p. 55, Fig. 13]. Figures 4.1(b) and 4.1(c) are recomposed from parts of drawings taken from the same source.

Table 4.2. Examples of Six Symbolic Representations of Odd and Even Multivectors in V^3.

	Appl. Diff. Geom.	Extnd. Grass. Algebra	William's Twisted Notation	Brackets		
				Extremum	Correlated	Native
Even	1	1	1	$\llparenthesis 1, 1 \rrparenthesis$	$\langle\!\langle 1, 1 \rangle\!\rangle$	$\llbracket 1, 1 \rrbracket$
	$\mathbf{e}_1$	$\mathbf{e}_1$	$\mathbf{e}_1$	$\llparenthesis 1, \mathbf{e}_1 \rrparenthesis$	$\langle\!\langle \mathbf{e}_1, \mathbf{e}_1 \rangle\!\rangle$	$\llbracket \mathbf{e}_1, \mathbf{e}_1 \rrbracket$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$\mathbf{e}_2$	$\llparenthesis 1, \mathbf{e}_2 \rrparenthesis$	$\langle\!\langle \mathbf{e}_2, \mathbf{e}_2 \rangle\!\rangle$	$\llbracket \mathbf{e}_2, \mathbf{e}_2 \rrbracket$
	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\llparenthesis 1, \mathbf{e}_{12} \rrparenthesis$	$\langle\!\langle \mathbf{e}_{12}, \mathbf{e}_{12} \rangle\!\rangle$	$\llbracket \mathbf{e}_{12}, \mathbf{e}_{12} \rrbracket$
	$\mathbf{e}_{13}$	$\mathbf{e}_{13}$	$\mathbf{e}_{13}$	$-\llparenthesis 1, \mathbf{e}_{31} \rrparenthesis$	$-\langle\!\langle \mathbf{e}_{31}, \mathbf{e}_{31} \rangle\!\rangle$	$-\llbracket \mathbf{e}_{31}, \mathbf{e}_{31} \rrbracket$
	Ω	$\mathbf{e}_{123}$	Ω	$\llparenthesis 1, \boldsymbol{\Omega} \rrparenthesis$	$\langle\!\langle \boldsymbol{\Omega}, \boldsymbol{\Omega} \rangle\!\rangle$	$\llbracket \boldsymbol{\Omega}, \boldsymbol{\Omega} \rrbracket$
Odd	$(1, \Omega)$	r	$\widehat{\Omega}$	$\llparenthesis \boldsymbol{\Omega}, 1 \rrparenthesis$	$\langle\!\langle \boldsymbol{\Omega}, 1 \rangle\!\rangle$	$\llbracket \boldsymbol{\Omega}, 1 \rrbracket$
	$(\mathbf{e}_1, \Omega)$	$r\mathbf{e}_1$	$\widehat{\mathbf{e}}_{23}$	$\llparenthesis \boldsymbol{\Omega}, \mathbf{e}_1 \rrparenthesis$	$\langle\!\langle \mathbf{e}_{23}, \mathbf{e}_1 \rangle\!\rangle$	$\llbracket \mathbf{e}_{23}, \mathbf{e}_1 \rrbracket$
	$(\mathbf{e}_2, \Omega)$	$r\mathbf{e}_2$	$\widehat{\mathbf{e}}_{31}$	$\llparenthesis \boldsymbol{\Omega}, \mathbf{e}_2 \rrparenthesis$	$\langle\!\langle \mathbf{e}_{31}, \mathbf{e}_2 \rangle\!\rangle$	$\llbracket \mathbf{e}_{31}, \mathbf{e}_2 \rrbracket$
	$(\mathbf{e}_{12}, \Omega)$	$r\mathbf{e}_{12}$	$\widehat{\mathbf{e}}_3$	$\llparenthesis \boldsymbol{\Omega}, \mathbf{e}_{12} \rrparenthesis$	$\langle\!\langle \mathbf{e}_3, \mathbf{e}_{12} \rangle\!\rangle$	$\llbracket \mathbf{e}_3, \mathbf{e}_{12} \rrbracket$
	$(\mathbf{e}_{13}, \Omega)$	$r\mathbf{e}_{13}$	$-\widehat{\mathbf{e}}_2$	$-\llparenthesis \boldsymbol{\Omega}, \mathbf{e}_{31} \rrparenthesis$	$-\langle\!\langle \mathbf{e}_2, \mathbf{e}_{31} \rangle\!\rangle$	$-\llbracket \mathbf{e}_2, \mathbf{e}_{31} \rrbracket$
	(Ω, Ω)	$r\mathbf{e}_{123}$	$\widehat{1}$	$\llparenthesis \boldsymbol{\Omega}, \boldsymbol{\Omega} \rrparenthesis$	$\langle\!\langle 1, \boldsymbol{\Omega} \rangle\!\rangle$	$\llbracket 1, \boldsymbol{\Omega} \rrbracket$

Need to say: the two sign conventions in defining Clifford algebras.

Need to say: We use a Clifford algebra as a formal calculating device. Similar to the way Kähler-Atiyah algebra is used.

We are about to define the product of the $\mathcal{OC}$ algebra, which we denote with a circled circle as in $\mathbf{a} \circledcirc \mathbf{b}$, by listing the laws that we require it to satisfy. These laws postulate the role of the $\mathcal{OC}$ product in

(1) interconverting the extremum bracket and the correlated bracket and
(2) calculating the exterior product of odd quantities.

The $\mathcal{OC}$ algebra is a Clifford-like algebra. We may define a Clifford-like algebra roughly as an algebra that is derived from a Clifford algebra by giving a set of rules that specify an additional sign (a factor of ± 1) to be attached to the Clifford product of two elements. This new product with a modified sign is the Clifford-like product. Properties, such as associativity, of the original Clifford algebra may be modified or annulled in any given Clifford-like algebra depending on the rule determining the added sign. A definition of a Clifford-like algebra based on the *maximally-graded* version of a Clifford algebra may be found in Reference [123, pp. 284 f.] or [83]. Later, in Section VIII, we will define the $\mathcal{OC}$ algebra by giving a sign rule of the type just discussed.

Our version of a Clifford-like algebra, the $\mathcal{OC}$ algebra inherits the scalar product (a nondegenerate, symmetric, positive-definite bilinear form $B(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in V^n$) associated with the quadratic form used to define the Clifford algebra. But we are not assuming that, in general this V^n comes to us with this or any other scaler product. Indeed, any such scalar product that the V of $\overline{\bigwedge} V^n$ may already have is irrelevant to the manipulations we perform with the $\mathcal{OC}$ product because we use the $\mathcal{OC}$ product only as a calculating tool operating on the gsign and gauge of the correlated and extremum brackets. We apply a tool, the $\mathcal{OC}$ algebra, associated with a metric to manipulate representations, the correlated and extremum brackets, of a space, V of $\overline{\bigwedge} V^n$, that, in general, is nonmetric. In this Section and always when used as a calculating tool in the sense just mentioned the $\mathcal{OC}$ algebra comes equipped with the Euclidean metric. The results of this Section will lead us toward a general axiomatic formulation of the $\mathcal{OC}$ algebra associated with a general, not necessarily Euclidean, metric.

We assume that V^3 has been assigned the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ with associated basis trivector (volume multivector) $\mathbf{\Omega} = \mathbf{e}_{xyz} = \mathbf{e}_x \wedge \mathbf{e}_y \wedge \mathbf{e}_z$ where as before, we may indicate the exterior product $\wedge$ by juxtaposition when convenient. Recall that we use the correlated and extremum brackets as bases to represent directed multivectors. Therefore both positions in either form may only contain a single even basis multivector, the unit scalar 1, or the negative of these, but not general linear combinations formed from the aforementioned. Throughout the following these basis multivectors are symbolized by uppercase italic letters as, for example, A.

The last task before we begin to derive rules of calculation in the $\mathcal{OC}$ algebra is to define two functions applicable to single basis multivectors or their negatives. Let $\mathscr{B}^\wedge$ be the set of 2^n basis multivectors for $\bigwedge V^n$.

In the following the index I_A in $\mathbf{e}_{I_A}$ is a multi-index, so that, for example in three dimensions I_A might expand to 12, thus giving $\mathbf{e}_{I_A} = \mathbf{e}_{12}$. We define the *sign function relative to a set of basis multivectors* $\mathscr{B}^\wedge$, written as $\operatorname{sgn}_{\mathscr{B}^\wedge} A$, or simply $\operatorname{sgn} A$ when it is clear what function and basis is meant, so that, $\operatorname{sgn}_{\mathscr{B}^\wedge} A = \lambda$ for all A such that $A = \lambda \mathbf{e}_{I_A}$ with $\lambda = \pm 1$ and $\mathbf{e}_{I_A} \in \mathscr{B}^\wedge$.

Complementarily, we define the *absolute value relative to a set of basis multivectors* $\mathscr{B}^\wedge$, written $|A|_{\mathscr{B}^\wedge}$, or simply $|A|$ when it is clear what function and basis is meant, so that, $|A|_{\mathscr{B}^\wedge} = \mathbf{e}_{I_A}$ for all A such that $A = \lambda \mathbf{e}_{I_A}$ with $\lambda = \pm 1$ and $\mathbf{e}_{I_A} \in \mathscr{B}^\wedge$.

Now we are ready to derive some fundamental laws for the $\mathcal{OC}$ product. Consider first the following rules that we postulate for interconversion between the extremum and correlated brackets.

We perform the following cyclic interconversion of brackets,

$$\text{extremum} \;\rightarrow\; \text{correlated} \;\rightarrow\; \text{extremum},$$

to obtain the results in Table 4.4.

Table 4.3. Interconversion Rules for Extremum and Correlated Bracket

	Extremum $\to$ Correlated	Correlated $\to$ Extremum
Even	$(\!(1, A)\!) \to \langle\!\langle A \circledcirc 1, A\rangle\!\rangle = \langle\!\langle A, A\rangle\!\rangle$	$\langle\!\langle C, D\rangle\!\rangle \to (\!(C \circledcirc D, D)\!)$
Odd	$(\!(\mathbf{\Omega}, A)\!) \to \langle\!\langle A \circledcirc \mathbf{\Omega}, A\rangle\!\rangle$	$\langle\!\langle C, D\rangle\!\rangle \to (\!(C \circledcirc D, D)\!)$

Table 4.4. Cyclic Interconversion of Extremum and Correlated Brackets

	Extremum	$\to$	Correlated	$\to$	Extremum	Extremum
Even	$(\!(1, A)\!)$	$\to$	$\langle\!\langle A \circledcirc 1, A\rangle\!\rangle$	$\to$	$(\!((A \circledcirc 1) \circledcirc A, A)\!) = \cdots$	$\cdots = \qquad (\!(1, A)\!)$
Odd	$(\!(\mathbf{\Omega}, A)\!)$	$\to$	$\langle\!\langle A \circledcirc \mathbf{\Omega}, A\rangle\!\rangle$	$\to$	$(\!((A \circledcirc \mathbf{\Omega}) \circledcirc A, A)\!) = \cdots$	$\cdots = (\!((A \circledcirc \mathbf{\Omega}) \wedge A, A)\!) = (\!(\mathbf{\Omega}, A)\!)$

The rules of Table 4.3 and the results of Table 4.4 imply the following laws for the orientation congruent product:

$$A \circledcirc 1 = A \wedge 1 = A, \quad \text{Right identity} \tag{4.13a}$$

$$A \circledcirc A = 1, \quad \text{Selfinverse} \tag{4.13b}$$

$$(A \circledcirc \mathbf{\Omega}) \circledcirc A = \mathbf{\Omega}, \quad \text{Left coinverse} \tag{4.13c}$$

$$(A \circledcirc \mathbf{\Omega}) \wedge A = \mathbf{\Omega}. \quad \text{Left exterior coinverse} \tag{4.13d}$$

Consider next the following rules for exterior products between odd and even multivectors expressed in extremum and correlated brackets. The last two bracket expressions of the following rules are related by the interconversion rules:

$$\text{Ext.} \quad (\!(\mathbf{\Omega}, A)\!) \wedge (\!(\mathbf{\Omega}, B)\!) = (\!(\mathbf{\Omega} \circledcirc \mathbf{\Omega}, A \wedge B)\!) = (\!(1, A \wedge B)\!),$$

$$\text{Cor.} \quad \langle\!\langle A \circledcirc \mathbf{\Omega}, A\rangle\!\rangle \wedge \langle\!\langle B \circledcirc \mathbf{\Omega}, B\rangle\!\rangle = \langle\!\langle (A \circledcirc \mathbf{\Omega}) \circledcirc (B \circledcirc \mathbf{\Omega}), A \wedge B\rangle\!\rangle = \langle\!\langle A \wedge B, A \wedge B\rangle\!\rangle.$$

These rules imply the following laws for the orientation congruent product where part (b) of Equation (4.14) is the *extension* of part (a) to include the case $A \wedge B = 0$:

$$(A \circledcirc \mathbf{\Omega}) \circledcirc (B \circledcirc \mathbf{\Omega}) = \begin{cases} \text{(a)} \quad A \wedge B, & \text{if } A \wedge B \neq 0, \qquad \text{Right } \mathbf{\Omega}\text{-complement} \\ \text{(b)} \quad A \circledcirc B, & \text{always.} \qquad \text{cancellation} \end{cases} \tag{4.14}$$

Finally, consider the following pairs of rules for exterior products between odd and even multivectors (and vice-versa) expressed in extremum and correlated brackets. Again, the last two bracket expressions in each pair of the following rules are related by the interconversion rules:

$$\text{Ext.} \quad (\!(\mathbf{\Omega}, A)\!) \wedge (\!(1, B)\!) = (\!(\mathbf{\Omega}, A \wedge B)\!),$$

$$\text{Cor.} \quad \langle\!\langle A \circledcirc \mathbf{\Omega}, A\rangle\!\rangle \wedge \langle\!\langle B, B\rangle\!\rangle = \langle\!\langle (A \circledcirc \mathbf{\Omega}) \circledcirc B, A \wedge B\rangle\!\rangle = \langle\!\langle (A \wedge B) \circledcirc \mathbf{\Omega}, A \wedge B\rangle\!\rangle,$$

$$\text{Ext.} \quad (\!(1, A)\!) \wedge (\!(\mathbf{\Omega}, B)\!) = (\!(\mathbf{\Omega}, A \wedge B)\!),$$

$$\text{Cor.} \quad \langle\!\langle A, A\rangle\!\rangle \wedge \langle\!\langle B \circledcirc \mathbf{\Omega}, B\rangle\!\rangle = \langle\!\langle A \circledcirc (B \circledcirc \mathbf{\Omega}), A \wedge B\rangle\!\rangle = \langle\!\langle (A \wedge B) \circledcirc \mathbf{\Omega}, A \wedge B\rangle\!\rangle.$$

These rules imply the following laws for the orientation congruent product where the combination of parts (a) and (d) of Equation (4.15) and that of parts (b) and (d) are the *extensions* of the combination of parts (a) and (c) of Equation (4.15) and that of parts (b) and (c), respectively:

Generalized commutativity of right $\mathbf{\Omega}$-complementation

$$\left.\begin{array}{ll}\text{(a)} & (A \circledcirc \mathbf{\Omega}) \circledcirc B \\ \text{(b)} & A \circledcirc (B \circledcirc \mathbf{\Omega})\end{array}\right\} = \begin{cases}\text{(c)} \quad (A \wedge B) \circledcirc \mathbf{\Omega}, & \text{if } A \wedge B \neq 0, \\ \text{(d)} \quad (A \circledcirc B) \circledcirc \mathbf{\Omega}, & \text{always.}\end{cases} \tag{4.15}$$

Let us introduce a more compact notation for the left and right $\mathbf{\Omega}$-complements using a prefixed or postfixed superscript uppercase omega as

$$\begin{array}{rl}{}^{\mathbf{\Omega}}A := \mathbf{\Omega} \circledcirc A, & \text{Left } \mathbf{\Omega}\text{-complementation} \\ A^{\mathbf{\Omega}} := A \circledcirc \mathbf{\Omega}. & \text{Right } \mathbf{\Omega}\text{-complementation}\end{array} \tag{4.16}$$

We call these operations left and right *counit complementation*, or just left and right *complementation* for short. To reduce the clutter of parentheses we give them precedence over orientation congruent, Clifford, and exterior product multiplications.

Now we may gather together the fundamental laws of Equations (4.13, 4.14, and 4.15) and rewrite them using this notation as follows:

$$A \circledcirc 1 = A, \quad \text{Right identity} \tag{4.13a$'$}$$

$$A \circledcirc A = 1, \quad \text{Selfinverse} \tag{4.13b$'$}$$

$$A^{\mathbf{\Omega}} \circledcirc A = \mathbf{\Omega}, \quad \text{Left coinverse} \tag{4.13c$'$}$$

$$A^{\mathbf{\Omega}} \wedge A = \mathbf{\Omega}; \quad \text{Left exterior coinverse} \tag{4.13d$'$}$$

$$A^{\mathbf{\Omega}} \circledcirc B^{\mathbf{\Omega}} = \begin{cases}\text{(a)} \quad A \wedge B, & \text{if } A \wedge B \neq 0, \\ \text{(b)} \quad A \circledcirc B, & \text{always;}\end{cases} \qquad \begin{array}{l}\text{Right } \mathbf{\Omega}\text{-complement} \\ \text{cancellation,}\end{array} \tag{4.14$'$}$$

(4.15′)

$$\left.\begin{array}{ll}\text{(a)} & A^{\mathbf{\Omega}} \circledcirc B \\ \text{(b)} & A \circledcirc B^{\mathbf{\Omega}}\end{array}\right\} = \begin{cases}\text{(c)} \quad (A \wedge B)^{\mathbf{\Omega}}, & \text{if } A \wedge B \neq 0, \\ \text{(d)} \quad (A \circledcirc B)^{\mathbf{\Omega}}, & \text{always.}\end{cases} \qquad \begin{array}{l}\text{Generalized commu-} \\ \text{tativity of right} \\ \mathbf{\Omega}\text{-complementation}\end{array}$$

In the multiplication tables for $\mathcal{C}\ell_n$ and $\mathcal{OC}_n$ that are presented as Tables 4.5 and 4.6 we have adopted the convention of symbolizing the basis n-vector $\mathbf{e}_{12\dots n} := \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n$ by $\boldsymbol{I}$ when it is involved in a Clifford product of some Clifford algebra derived from an n-dimensional base space, or by $\mathbf{\Omega}$ when it is involved in an orientation congruent product of some orientation congruent algebra derived from an odd-dimensional base space.

TABLE 4.5. The Multiplication Table for the Clifford Algebra $\mathcal{C}\ell_3$. The factors are in reflected, complementary grade order with indices in orientation congruent order.

		b							
	$a \circ b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{I}$
a	1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{I}$
	$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$\mathbf{I}$	$\mathbf{e}_{23}$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_1$	$\mathbf{I}$	$\mathbf{e}_3$	$\mathbf{e}_{31}$
	$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\mathbf{I}$	$\mathbf{e}_1$	$-\mathbf{e}_2$	$\mathbf{e}_{12}$
	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{I}$	-1	$\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_3$
	$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$\mathbf{I}$	$-\mathbf{e}_1$	$-\mathbf{e}_{23}$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_2$
	$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\mathbf{I}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	-1	$-\mathbf{e}_1$
	$\mathbf{I}$	$\mathbf{I}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	-1

TABLE 4.6. The Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_3$. The factors and indices are ordered as in Table 4.5 above. Red cell entries are signed oppositely to those in Table 4.5.

		b							
	$a \circledcirc b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{\Omega}$
a	1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{\Omega}$
	$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{\Omega}$	$\mathbf{e}_{23}$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$\mathbf{e}_1$	$\mathbf{\Omega}$	$-\mathbf{e}_3$	$\mathbf{e}_{31}$
	$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\mathbf{\Omega}$	$-\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$\mathbf{\Omega}$	1	$-\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$
	$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$-\mathbf{e}_3$	$\mathbf{\Omega}$	$\mathbf{e}_1$	$\mathbf{e}_{23}$	1	$-\mathbf{e}_{12}$	$\mathbf{e}_2$
	$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\mathbf{\Omega}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_{31}$	$\mathbf{e}_{12}$	1	$\mathbf{e}_1$
	$\mathbf{\Omega}$	$\mathbf{\Omega}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_1$	1

5. An Axiom System for the Orientation Congruent Algebra

> Grassmann's one great contribution to mathematics is ... the definition of exterior algebra. He gave his entire life to understanding and developing this definition....
>
> Evil tongues whispered that there was really nothing new in Grassmann's algebra.... The standard objection was ..., "What can you prove with exterior algebra that you cannot prove without it?" Whenever you hear this question ... you are likely to be in the presence of something important.... A proper retort might be: "You are right. There is nothing in yesterday's mathematics that you can prove with exterior algebra that could not also be proved without it. Exterior algebra is not meant to prove old facts, it is meant to disclose a new world. Disclosing new worlds is as worthwhile a mathematical enterprise as proving old conjectures."
>
> Gian-Carlo Rota [152, pp. 46–48]

In this section we first discuss the nondegenerate quadratic form $Q_{p,q}$, defining terms and notations for it, and then, later, the two algebras of our interest associated with it. We then present a deductive foundation for the Clifford algebra $\mathcal{C}\ell_{p,q}$ of a nondegenerate quadratic form $Q_{p,q}$ in terms of *generators* of and *relations* on its elements (a *GR* axiom system for short). Next we give a similar axiomatic formulation for the orientation congruent algebra $\mathcal{OC}_{p,q}$ that is derived from the one for the corresponding Clifford algebra $\mathcal{C}\ell_{p,q}$ by modifying two of its axioms and adding two new axioms. Although, in this section, we give only GR axiom sets, in this section's penultimate subsection we discuss some alternative axiomatic approaches. Finally, the last subsection of this section presents the multiplication tables for some low order Clifford and orientation congruent algebras.

5.1. **Geometric, Clifford, and Orientation Congruent Algebras.** Hestenes, Li, and Rockwood in their book contribution [96, p. 2] give a similar, but streamlined treatment of geometric algebra.

What is geometric algebra as practiced by David Hestenes and his colleges that aims to be a "universal geometric algebra" and "a unified language for mathematics and physics" [93, 92, 95]? Briefly, geometric algebra is Hestenes presentation of a Kähler-Atiyah algebra together with a meta-algebraical, usually geometric, interpretation.

What is geometric algebra à la Hestenes? Geometric algebra is the Kähler-Atiyah algebra [20, p. 86], [78], [79], [145, pp. 5597 f.] interpreted geometrically. In fact, many different geometric interpretations of geometric algebra have been explored in the literature. Intuitively, the Kähler-Atiyah algebra is a kind of abstract, basis-independent, super matrix algebra. Matrix algebra represents the linear transformations of linear algebra in a basis. A linear transformation on a linear space is an endomorphism—a map from a set to itself. Linear algebra treats vector spaces. Multilinear algebra treats tensor products of vector spaces, their tensor algebras, and the abstract algebras derived from them such as the exterior algebra and Clifford algebra. The set of elements of Clifford algebra in their operator form is isomorphic to the set of all linear endomorphisms on the linear space of exterior algebra. Geometric algebra as a Kähler-Atiyah algebra has a metric associated with its Clifford algebra part. If the *background metric* of geometric algebra is taken as the Euclidean one in an orthonormal basis, interpretational metrics may be treated as WHAT see Hestenes page.

As practiced by David Hestenes and his followers geometric algebra is a way to compute as with matrices but without necessarily invoking arrays of numbers as the expression of objects in a coordinate basis. Just as a production company provides a troupe of stock actors ready to assume any role according to a playwright's script, matrix theory provides us with a matrices ready to model all sorts of mathematical objects such as linear transformations bilinear forms on a vector space (metrics), tensors, or the points, lines, and planes of Euclidean, affine, or projective geometry, to name just a few, all according to our ingenuity and imagination.

Hestenes' drive to make geometric algebra and calculus a universal language for mathematics and physics rivals that of the utopian constructors of Esperanto. Unfortunately, his efforts suffer the same ill effects of scientific, rather than general, linguistic parochialism. Another disincentive to adopting the powerful techniques of geometric algebra and calculus is that they are subtly complicated. The meaning of a word in Chinese depends not only on its basic sound but on the nuances of tonal inflection. Similarly, in geometric algebra the meaning of a given form is chosen from among many possible interpretations.

Indeed, in this work we do not fully embrace the geometric algebra and calculus of Hestenes' expansive vision. In particular, we work here not with geometric calculus, but with classical differential geometry. However, we do borrow geometric algebra's vocabulary, its axiomatic foundations, its general results, and, its particular results as applied to projective geometry.

We formulate the orientation congruent algebra by modifying the axiomatic foundation of geometric algebra given by Hestenes and Sobczyk [97]. We choose this axiomatic basis for the $\mathcal{OC}$ algebra for three reasons. First, the Hestenes-Sobczyk axioms start from the familiar concept of a vector space. Second, by changing a few key axioms the Hestenes-Sobczyk formulation of geometric algebra can be easily modified to define the orientation congruent algebra, thus allowing the parallel development of both algebras. This is possible, in part, because like the geometric algebra, which is a Kähler-Atiyah algebra containing both the Clifford product and the outer (exterior) product, the $\mathcal{OC}$ algebra is a generalized Kähler-Atiyah algebra containing both the orientation congruent product and the outer product. Third, the modified Hestenes-Sobczyk axioms permit an independent characterization of the orientation congruent algebra as a Clifford-like algebra in Section 6 using the sign factor function σ. Such an independent characterization is not possible using the other common definition of a Clifford algebra as the quotient of an abstract tensor algebra by an ideal because the tensor algebra is inherently associative and the orientation congruent algebra is nonassociative. I know of only one other route to the $\mathcal{OC}$ algebra that is independent of its Clifford-likeness. However, that route requires the still more abstract concept of a Hopf algebra which is unsuitable for this introductory paper.

Because some of the conceptual bases of the Clifford and orientation congruent algebras differ sharply, we define these two algebras by separate but parallel axiom systems. Most of the axioms in the two systems are identical and are not repeated. However, in this parallel presentation, the key axioms differing for the two algebras are more easily compared and contrasted. In the end, though, a final axiom set allows the two algebras to merge into a generalized Kähler-Atiyah algebra containing three products: the Clifford, orientation congruent, and outer (exterior) product. For the most part to avoid the ungainly phrase *Clifford-orientation congruent algebra*, we simply refer to this merged algebra by the abbreviated name $\mathcal{CO}$ *algebra*.

The initial Hestenes-Sobczyk axioms describe a Kähler-Atiyah algebra over a base space $V := \langle \mathcal{C}\ell_{p,q} \rangle_1$ of 1-vectors without defining the concept of the dimension n of V, but with the quadratic form Q associated with V being Euclidean, i.e. $Q(\mathbf{v}) > \mathbf{0}$ for all $\mathbf{v} \neq \mathbf{0} \in \mathbf{V}$. Hestenes and Sobczyk later [97, pp. 16–20] define the dimension n by introducing a top-dimensional unit-magnitude n-blade, the *pseudoscalar* $\mathbf{I_n}$ through the implicit axiom that there exists a unit-magnitude n-blade $\mathbf{I_n}$ such that $n \geq 1$ (since vectors exist) and

$$V = \{\mathbf{v} \mid \mathbf{v} \wedge \mathbf{I_n} = \mathbf{0}\}.$$

Then the dimension n (possibly infinite) of the algebra's base space V is defined to be the grade n of $\mathbf{I_n}$. Also the general concept of a not necessarily Euclidean (pseudo-Euclidean) metric is introduced by Hestenes and Sobczyk as either "built into" V through the geometric algebra equivalent of Definitions 5.1 and 5.2 below for the signature (p, q), $n = p + q$, of the quadratic form Q associated with V [97, pp. 41–43, 102–111], or, more powerfully, as an "auxiliary" linear transformation and its equivalent bilinear form and modified inner product [97, pp. 96–102].

In contrast to the approach of Hestenes and Sobczyk, it is more convenient for us to use axioms for the Clifford and orientation congruent algebras that are modified versions of theirs, so that, from the start, our axioms define the base vector space to have the finite dimension, n. (Although, for the orientation congruent algebra, if n is even, we have to consider the extension of the base space dimension to $n + 1$.) We do this because, in this paper, we use the orientation congruent algebra to describe only the *oriented vector subspaces* of a vector space with a *fixed, finite* dimension n (extensible to $n + 1$, if n is even). For unoriented subspaces the analogous description is known to mathematicians as the Plücker embedding [85, ch. 11]. By representing vector spaces as exterior products of vectors, the Plücker embedding allows us to calculate with vector spaces—a technique first employed by Hermann Günther Grassmann in his pioneering works of the mid-19th century [80, 81].

5.2. The Nondegenerate Quadratic Form $Q_{p,q}$ and Associated Algebras.

Let us define the parallel relationships of the notations Q, $Q_{p,q}$, and Q_n; $\mathcal{C}\ell(Q)$, $\mathcal{C}\ell_{p,q}$, and $\mathcal{C}\ell_n$; and $\mathcal{OC}(Q)$, $\mathcal{OC}_{p,q}$, and $\mathcal{OC}_n$. Here n, p, and q are integers such that $n \geq 1$, and $p, q \geq 0$ with $p \geq 1$ or $q \geq 1$. First, we need the notions of a general *quadratic form* Q and its associated *symmetric bilinear form* B_Q.[15]

Definition 5.1 (Quadratic Form and Associated Symmetric Bilinear Form)**.**
This definition is taken from Fauser [67, p. 3]. A *quadratic form* on a vector space

[15] A map with two arguments such that $B \colon U \times V \to W$, where U, V, and W are vector spaces over $\mathbb{R}$, is said to be *bilinear* if and only if it is linear in both of its arguments. That is, $B(x + y, z) = B(x, z) + B(y, z)$, $B(x, y + z) = B(x, y) + B(x, z)$, and $B(\alpha x, \beta y) = \alpha\beta\ B(x, y)$ for all $\alpha, \beta \in \mathbb{R}$. The *form* part of its name means that for B_Q we have $W = \mathbb{R}$ in the definition of bilinearity just given. Also the word *symmetric* implies that $U = V$, since it means that $B_Q(x, y) = B_Q(y, x)$.

V^n over $\mathbb{R}$ is a map $Q\colon V^n \to \mathbb{R}$ such that

$$Q(\alpha x) = \alpha^2 Q(x) \text{ for all } \alpha \in \mathbb{R} \text{ and } x \in V, \text{ and} \tag{5.1a}$$

$$B_Q(x,y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y)) \text{ for all } x, y \in V^n, \tag{5.1b}$$

where $B_Q\colon V^n \times V^n \to \mathbb{R}$ is the *symmetric bilinear form* associated with Q by the *polarization relation* given by equation (5.1b).

Definition 5.2 (Nondegenerate Quadratic Form of Signature (p,q))**.**
A quadratic form for the n-dimensional vector space V^n such that $Q(x) \neq 0$ for all $x \in V^n$ is said to be *nondegenerate.* Let Q be a nondegenerate quadratic form on V^n. If there exists an indexed set of *pairwise orthogonal* vectors $\{\, \mathbf{e}_1, \ldots, \mathbf{e}_p, \mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+q} \,\}$ for V^n (that is, $B_Q(\mathbf{e}^i, \mathbf{e}^j) = 0$ for all $i \neq j$) such that for all $\mathbf{e}_i$

$$\begin{aligned} Q(\mathbf{e}_i) &> 0, \quad \text{for} \qquad 1 \leq i \leq p, \quad \text{and} \\ Q(\mathbf{e}_i) &< 0, \quad \text{for} \;\; p+1 \leq i \leq p+q = n, \end{aligned}$$

we say that Q is of *signature*[16] (p,q) and we may write $Q_{p,q}$ to signify this. If $q = 0$, we have $p = n$; whereupon we say Q is of *positive signature* n and we may write Q_n to indicate it. Also, if $p = 0$, we have $q = n$; whereupon we say Q is of *negative signature* n and we may write $Q_{0,n}$ to indicate that.

We may also represent that a nondegenerate quadratic form $Q_{p,q}$, Q_n, or $Q_{0,n}$ exists for the vector space V^n by writing $V^{p,q}$, $V^{n,0}$, or $V^{0,n}$, respectively.[17] We symbolize the corresponding Clifford algebras by $\mathcal{C}\ell_{p,q}$, $\mathcal{C}\ell_n$, and $\mathcal{C}\ell_{0,n}$; and the corresponding orientation congruent algebras by $\mathcal{OC}_{p,q}$, $\mathcal{OC}_n$, and $\mathcal{OC}_{0,n}$.[18] When discussing the Clifford or orientation congruent algebra of a general quadratic form Q, or when the signature (p,q) of Q is understood from context, we may also write $\mathcal{C}\ell(Q)$ or $\mathcal{OC}(Q)$, respectively. And when referring to the nondegenerate quadratic form of signature (p,q) associated with the Clifford algebra $\mathcal{C}\ell_{p,q}$, we will usually write simply Q instead of $Q_{p,q}$.

5.3. **MHS Axioms for the Clifford Algebra $\boldsymbol{\mathcal{C}\ell_{p,q}}$.** Before we give a set of axioms for the orientation congruent algebra of a quadratic form, $\mathcal{OC}_{p,q}$, we first introduce a set of axioms for the Clifford algebra of a quadratic form, $\mathcal{C}\ell_{p,q}$, adapted from Hestenes and Sobczyk's presentation. This modified Hestenes-Sobczyk (*MHS*) axiom set for $\mathcal{C}\ell_{p,q}$ comprises a long list of 26 axioms divided into seven sets. Later in this section, by altering this axiomatic formulation for $\mathcal{C}\ell_{p,q}$, we obtain a modified Hestenes-Sobczyk axiom set for $\mathcal{OC}_{p,q}$ which comprises an even longer list of 28 axioms divided into eight sets. A final set of two axioms are provided which allow the outer (exterior), Clifford, and orientation congruent products to coexist in an extended Kähler-Atiyah algebra which we call the *Clifford-orientation congruent algebra* of a quadratic form, $\mathcal{CO}_{p,q}$.

[16]This (p,q) is, of course, the physicist's signature, not $s = p - q$, the mathematician's version.

[17]Note that we do not use V^n as a brief form of $V^{n,0}$, as we do for the corresponding notations for the Clifford and orientation congruent algebras, because we reserve V^n to indicate the vector space of dimension n that does not necessarily have a quadratic form associated with it.

[18]The trivial *algebras* $\mathcal{C}\ell_{0,0}$ and $\mathcal{OC}_{0,0}$ also exist, but are not associated with a quadratic form since they are isomorphic with $\mathbb{R}$.

These axioms for $\mathcal{C}\ell_{p,q}$ were adapted primarily from those of Hestenes and Sobczyk's book [97, pp. 3 ff.] and Perwass's publications [139, pp. 12 f.], [140, pp. 22–24]. Shaw's book [166, pp. 6,9] was also consulted for the vector space properties postulated in Axiom Set I. The style and typography in which these axioms are presented is modeled after that used by Perwass [ibid.]. However, differing from Hestenes and Sobczyk's treatment, ours is somewhat simplified in the following ways:

(1) we construct one algebra with a base 1-vector space $V^{p,q}$ that has a fixed, finite dimension $n = p + q$ large enough to contain all subspaces of interest, rather than construct an infinite lattice of nested subalgebras, one for each pair of nonnegative integers (p, q) at least one of which is positive;
(2) we introduce the base space $V^{p,q}$ and its associated quadratic form $Q_{p,q}$ first independent of their derivation from the axioms;
(3) we restrict this quadratic form $Q_{p,q}$ to be nondegenerate.

Following the conventions of Hestenes and Sobczyk, hereafter the term *multivector* is used for any (not necessarily homogeneous) element of the Clifford algebra $\mathcal{C}\ell_{p,q}$ (or the orientation congruent algebra $\mathcal{OC}_{p,q}$) including those containing a scalar or vector component, the term *grade* (of a multivector) is used for the concept similar to the one usually referred to as *degree*, *step*, or *rank*, and a superscript dagger, as in $A^\dagger$, is used to represent the *reversion* (or *main anti-automorphism* [51, p. 30]) operation of a Clifford algebra.

Although Hestenes and Sobczyk, as well as many other authors, indicate Clifford multiplication by juxtaposition, we prefer to distinguish between it and orientation congruent multiplication by giving each its own symbol: an open dot (a very small circle) $\circ$ for the Clifford product and a circled open dot $\circledcirc$ for the orientation congruent product. Our use of an open dot for the Clifford product follows the practice of Rota and Stein in their paper [153], where they also use the term *circle product*.

For some comments on the nature of the geometric algebra and calculus practiced by Hestenes and his followers see the reviews by R. J. Plymen [141] and A. Crumeyrolle [50] of *Clifford Algebra to Geometric Calculus*. More insight is found in the paper by Aragón et al. [3] and the review of it by W. A. Rodrigues, Jr. [149]. The perspective of the computer graphics researcher applying geometric algebra techniques for geometric modeling is given by Leo Dorst's concise *Geometric Algebra FAQ* [62].

See Lounesto's book [123, pp. 190–192] or his book chapter [124, pp. 26–27]. Chapters 14, 21, and 22 of Lounesto's book [123] give several other definitions of a Clifford algebra.

Applied works commonly use a long set of axioms similar to those we give next to define the Clifford algebra $\mathcal{C}\ell_{p,q}$; however, usually their authors do not also mention the mathematically sophisticated refinement of condition (3).

For a more detailed discussion of universality under the name *unique factorization property*, and in the context of the tensor product of vector spaces, see Shaw [167, pp. 274–277]. For a specifically Clifford algebraic discussion see Gilbert and Murray's book [77, pp. 12–17] or the brief treatment in Perwass's thesis [139, p. 18]. For the related category theoretic formulation of Clifford algebras see Lounesto's book chapter [124, pp. 26–29]. Lastly, for readers of German, the discussion in Jung's thesis [109, app. A.4] appears to be good.

For reference and completeness we next give a modified Hestenes-Sobczyk axiomatic definition for the Clifford algebra of a nondegenerate quadratic form $\mathcal{C}\ell_{p,q}$ in a list of 26 axioms divided into seven sets.

This list starts with two sets of axioms which are the standard vector space axioms; however, now the vector space contains the multivectors in $\mathcal{C}\ell_{p,q}$ rather than just the vectors in the base space V^n. The first set of axioms given by Axiom Set I defines the properties of *multivector addition*; the second set given by Axiom Set II, the properties of two-sided *scalar multiplication.* Axiom Sets III through VII add the last 15 axioms that define the *algebraic* properties of *Clifford multiplication.*

Axiom III.2 below assumes that $\mathbb{R} \subseteq \mathcal{C}\ell_{p,q}$; that is, that scalars are multivectors. Similarly, Axiom VII.2 assumes that $V^n \subseteq \mathcal{C}\ell_{p,q}$; that is, that vectors are multivectors. However, in a more careful interpretation, one says that $\mathbb{R}$ and V^n are present in $\mathcal{C}\ell_{p,q}$ only as isomorphic images. The approach adopted here of identifying $\mathbb{R}$ and V^n with their images in $\mathcal{C}\ell_{p,q}$ creates redundancies in our axiom system that are discussed in detail in the footnotes. There we see that most of the axioms in the second set are subsumed and mirrored in those of the third set; scalar multiplication of multivectors will have become, after all, just Clifford multiplication by a scalar, and so, must be consistent with it.

The first set of axioms defines the set of multivectors, $\mathcal{C}\ell_{p,q}$, as an *abelian group* under the operation of multivector addition. The *group operation* is written as an addition sign $+$.

Axiom Set I. *Vector Space Addition of Multivectors.*
For all Clifford algebras $\mathcal{C}\ell_{p,q}$, there exists a binary operation called *multivector addition,* symbolized by the *addition sign* $+$, and which is said to produce the *sum* of two multivectors, such that for all $A, B, C \in \mathcal{C}\ell_{p,q}$

(I.1)	$A + B \in \mathcal{C}\ell_{p,q}$,	Closure of multivector addition
(I.2)	$A + B = B + A$,	Commutativity of multivector addition
(I.3)	$(A + B) + C = A + (B + C)$,	Associativity of multivector addition
(I.4)	$A + 0 = A$, and	Existence of an identity
(I.5)	$A + (-A) = 0$.	Existence of an inverse[19]

In the second axiom set and below the elements of $\mathbb{R}$ are a special group of multivectors called *scalars.* We usually denote scalars by lower case Greek letters.

Axiom Set II. *Vector Space Two-Sided Scalar Multiplication of Multivectors.*
For all Clifford algebras $\mathcal{C}\ell_{p,q}$, there exists a special subset of multivectors $\mathbb{R} \subseteq \mathcal{C}\ell_{p,q}$, the set of *scalars*, and a binary operation $\mathbb{R} \times \mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{p,q}$ and $\mathcal{C}\ell_{p,q} \times \mathbb{R} \to \mathcal{C}\ell_{p,q}$ called *scalar multiplication,* symbolized by *juxtaposition,* such that for all $A, B \in \mathcal{C}\ell_{p,q}$ and $\alpha, \beta \in \mathbb{R}$

(II.1)	$\alpha A, A\alpha \in \mathcal{C}\ell_{p,q}$,	Closure of scalar multiplication
(II.2)	$A\alpha = \alpha A$,	Commutativity of scalar multiplication[20]
(II.3)	$(\alpha\beta)A = \alpha(\beta A)$,	Associativity of left scalar multiplication
(II.4)	$1A = A$, and	Existence of a left identity

[19]This axiom is derivable from others in the two sets of vector space axioms and the field properties of $\mathbb{R}$ if we *define* $-A$ to be the result of the scalar multiplication $(-1)A$.

(II.5)	$(\alpha+\beta)A = \alpha A + \beta A,$ $\alpha(A+B) = \alpha A + \alpha B.$	Distributivity of left scalar multiplication over scalar and multivector addition

Definition 5.3 (Algebra over $\mathbb{R}$)**.**
An *algebra* over $\mathbb{R}$ is a vector space W together with a bilinear[21] binary operation, m, called the algebra's *product* or *multiplication,* such that $m\colon W \times W \to W$ as $m\colon (x,y) \mapsto m(x,y)$. Usually $m(x,y)$ is written in infixed notation as $x \textcircled{m}\, y$, where $\textcircled{m}$ is usually some more abstract symbol such as $\circ$. Sometimes a product is simply indicated by juxtaposing the multiplicands as in $x\,y$.

Adding the third set of axioms turns the vector space $\mathcal{C}\ell_{p,q}$ into a general (nonassociative) algebra over $\mathbb{R}$. This algebra inherits 1 from the vector space as its *unit* or *identity element* by Axioms II.4 and III.2.

Axiom Set III. *The $\mathcal{C}\ell_{p,q}$ Product: General Properties.*
For all Clifford algebras $\mathcal{C}\ell_{p,q}$, there exists an algebraic product called *Clifford multiplication*, symbolized by an *open dot* $\circ$*,* such that for all $A, B, C \in \mathcal{C}\ell_{p,q}$ and all $\alpha \in \mathbb{R}$

(III.1)	$A \circ B \in \mathcal{C}\ell_{p,q},$	Closure of Clifford multiplication
(III.2)	$\alpha \circ A = \alpha A,$ $A \circ \alpha = A\alpha$, and	Equality with left and right scalar multiplication[22]
(III.3)	$A \circ (B+C) = A \circ B + A \circ C,$ $A(B+C) \circ A = B \circ A + C \circ A.$	Left and right distributivity over multivector addition[23]

Implicit in the expressions of Axiom Set III is the usual parentheses-sparing convention of performing Clifford multiplications before performing multivector additions. Specifically, in Axiom III.3 this operator precedence rule is applied on the right sides of the equations.

The next set of axioms, the fourth, defines the fundamental properties of the grade projection operator in preparation for Axiom Sets V and VI.

Axiom Set IV. *Grade Projection of Multivectors.*
For all Clifford algebras $\mathcal{C}\ell_{p,q}$, there exists a projective operator on multivectors called *grade projection*, symbolized by *angular brackets with an integer subscript* as $\langle A\rangle_r$, that selects the *r-grade part* of multivectors and such that for all $A, B \in \mathcal{C}\ell_{p,q}$, all $\alpha \in \mathbb{R}$, and all $r, s \in \mathbb{Z}$

(IV.1)	$\langle A\rangle_r \in \mathcal{C}\ell_{p,q},$	Closure of grade projection

[20]Since $\mathbb{R}$ is a field, and thus has a *commutative* multiplication, it is not necessary to assume the existence of right scalar multiplication $A\alpha$ in Axiom II.1. Axiom II.2 may then be taken as a definition of right scalar multiplication as $A\alpha := \alpha A$. See Shaw's Remark (b) [166, p. 9].

[21]A binary operation is bilinear if and only if it is linear in both of its arguments. Bilinearity implies distributivity of the product over vector space addition. Nevertheless, we explicitly include the distributive property in the axioms. For a more general definition of bilinearity see Footnote 15.

[22]As mentioned above, we have assumed that scalars are multivectors $\mathbb{R} \subseteq \mathcal{C}\ell_{p,q}$. Therefore, the properties of scalar multiplication given in Axiom Set II are partially subsumed under those of Clifford multiplication given in this axiom set. In particular, this axiom and the one above it make Axiom II.1 redundant and it may be dropped.

[23]Axiom III.3, the distributivity of Clifford multiplication, with the help of Axiom III.2, implies the (now redundant) Axiom II.5, the distributivity of left scalar multiplication.

(IV.2)	$\langle A\rangle_r = 0,\ \text{ if } r < 0,$	Negative nullity of grade projection
(IV.3)	$\langle A + B\rangle_r = \langle A\rangle_r + \langle B\rangle_r,$	Additive linearity of grade projection
(IV.4)	$\langle \alpha A\rangle_r = \alpha\langle A\rangle_r,$	Left multiplicative linearity of grade projection
(IV.5)	$\langle\langle A\rangle_r\rangle_r = \langle A\rangle_r,$	Projectivity of grade projection
(IV.6)	$\langle\langle A\rangle_r\rangle_s = 0,\ \text{ if } r \neq s, \text{ and}$	Orthogonality of grade projection[24]
(IV.7)	$A = \sum_r \langle A\rangle_r.$	Grade decomposability of multivectors

The fifth set of axioms defines the grade of scalars as 0. It also defines a subset of the algebra's multivectors that is the underlying vector space of the algebra—the algebra's *base space* V^n, where $n = p + q$. The base space is a graded vector space [166, pp. 10 f.] which is *homogeneous*, that is, all its elements, the *vectors*, have the same grade, which, in this case, is 1. We usually write vectors as bold unitalicized lower case letters as in $\mathbf{a}$.

Axiom Set V. *Grades of Scalars and Vectors.*
For all Clifford algebras $\mathcal{C}\ell_{p,q}$, the grade projection of multivectors is such that

(V.1)	for all $\alpha \in \mathbb{R}$, $\langle\alpha\rangle_0 = \alpha$, and	Scalars have grade 0
(V.2)	there exists a set $V^n \in \mathcal{C}\ell_{p,q}$ such that for all $\mathbf{a} \in V^n$, $\langle\mathbf{a}\rangle_1 = \mathbf{a}$.	Vectors (of the base space V^n) exist as multivectors with grade 1

Before presenting the next axiom set we pause to make some fundamental definitions which are used throughout the sequel.

Definition 5.4 (Clifford r-Blade and Clifford r-Vector)**.**

(1) A Clifford *r-blade* is Clifford multivector, which we usually symbolize with a bold upper case unitalicized letter as in $\mathbf{A}$, such that it can be expressed as a *freely-parenthesized* Clifford multiproduct of r pairwise anticommuting vectors for some integer $2 \leq r \leq n$. That is, $\mathbf{A} = \mathbf{a}_1 \circ \cdots \circ \mathbf{a}_i \circ \cdots \circ \mathbf{a}_r$, with all groupings into binary products equal to each other, and $\mathbf{a}_i \circ \mathbf{a}_j = -\mathbf{a}_j \circ \mathbf{a}_i$ for all $i \neq j$. Note that we have used the convention that an unparenthesized multiproduct represents all parenthesizations of the multiproduct into binary products. Sometimes we write a Clifford r-blade with a subscript to indicate its grade as in $\mathbf{A}_r$. We write the set of all Clifford blades of the Clifford algebra $\mathcal{C}\ell_{p,q}$ as $\mathcal{CB}\ell_{p,q}$ and the set of all Clifford r-blades of $\mathcal{C}\ell_{p,q}$ as $\mathcal{CB}\ell^r_{p,q}$.

(2) We also define *1-blade* to mean *vector,* and *0-blade* to mean *scalar*. We interpret a multiproduct expression such as $\prod_{1\leq i\leq r} \mathbf{a}_i = \mathbf{a}_1 \circ \cdots \circ \mathbf{a}_i \circ \cdots \circ \mathbf{a}_r$ to be the single vector $\mathbf{a}_1$, when $r = 1$, and some scalar α, when $r = 0$.

(3) For any integer $0 \leq r \leq n$ all zero-magnitude r-blades are considered to be equivalent. Thus, 0 represents a blade of indeterminate grade, $0 \in \mathcal{B}\ell^r_{p,q}$ for all $0 \leq r \leq n$ [166, pp. 10–11], as well as indeterminate direction [26, pp. 296–297].

(4) A Clifford *r-vector*, also called a *homogeneous Clifford multivector of grade* r, is defined as a linear combination of Clifford r-blades. We often write a

[24]This axiom is due to Aragón et al. [3].

Clifford r-vector as an upper case italic letter with an attached subscript that indicates its grade as in A_r.

(5) The set or vector space of all Clifford r-vectors in the Clifford algebra $\mathcal{C}\ell_{p,q}$ may be written with a superscript r as $\mathcal{C}\ell^r_{p,q}$. Extending the grade projection operator to sets or vector spaces of multivectors in a natural way, we may also write $\mathcal{C}\ell^r_{p,q} = \langle \mathcal{C}\ell_{p,q} \rangle_r$. Then, the vector space of all multivectors in $\mathcal{C}\ell_{p,q}$ may be written as the direct sum of the vector spaces of all grades of multivectors: $\mathcal{C}\ell_{p,q} = \bigoplus_r \mathcal{C}\ell^r_{p,q} = \bigoplus_r \langle \mathcal{C}\ell_{p,q} \rangle_r$. Finally, the set or vector space of all (1-)vectors may alternatively be written as $V^n = \mathcal{C}\ell^1_{p,q}$.

The sixth set of axioms defines the grade of blades. It also makes the blades of the Clifford algebra $\mathcal{C}\ell_{p,q}$ its fundamental additive and multiplicative building blocks. Our use of the word *grade* here is somewhat at odds with its general use because, although we have assigned a grade to blades, it is the $\mathbb{Z}$-grade, isomorphic to the additive group of $\mathbb{Z}$, for the exterior algebra over V^n, not the $\mathbb{Z}_2$-grade, isomorphic to the $\mathbb{Z}_2$ additive group, which is the natural one for multivectors under Clifford multiplication.

This apparent discrepancy is resolved when we see that our generators and relations axiomatic formulation of Clifford algebras is equivalent to a Kähler-Atiyah algebra which allows us to use the set of all blades $\mathcal{B}\ell_{p,q}$, as isomorphic to the Grassmann algebra $\bigwedge V^n$. In any given basis $\mathscr{B}$ the set of all basis blades $\mathcal{B}\ell_{\mathscr{B}}$ is isomorphic to $\bigwedge \mathscr{B}$, the set of all exterior products in canonical form formed from the basis vectors, and allows us to calculate Clifford products in terms of exterior products. However, the isomorphism between $\mathcal{C}\ell_{p,q}$ and $\bigwedge V^n$ is only a vector space isomorphism, not an algebra isomorphism [51, p. 45]. Oziewicz has created a more elaborate theory under the name *Clifford algebras of multivectors* that explains this discrepancy [136], [68, pp. 25–27].

Axiom Set VI. *The $\mathcal{C}\ell_{p,q}$ Product: r-Blade and r-Vector Properties.*
For all Clifford algebras $\mathcal{C}\ell_{p,q}$, Clifford multiplication is such that for all $A \in \mathcal{C}\ell_{p,q}$ and $r \in \mathbb{Z}$

(VI.1) $\left\langle \prod_{1\le i\le r} \mathbf{a}_i \right\rangle_r = \prod_{1\le i\le r} \mathbf{a}_i$, for all $\mathbf{a}_i \in V^n$ such that $\mathbf{a}_i \circ \mathbf{a}_j = -\mathbf{a}_j \circ \mathbf{a}_i$ if $i \neq j$, and — r-Blades have grade r

(VI.2) there exists an $1 \le m \in \mathbb{Z}$ and a set of m r-blades $\{\mathbf{B}_i\}$ such that $\langle A \rangle_r = \sum_{1\le i\le m} \mathbf{B}_i$. — Blade decomposability of homogeneous multivectors

Next, we define the Clifford outer product of multivectors in terms of the Clifford algebra product. This definition interrupts the flow of the Clifford algebra axioms. However, in the axiom system for the orientation congruent algebra we are forced to introduce the corresponding definition for the outer product of multivectors before moving to the next axiom set. Therefore, I place the Clifford outer product definition here so that the two axiom systems are developed in parallel. I do not, however, repeat it as an orientation congruent axiom because the required modifications are trivial.

Definition 5.5 (Clifford Algebra Outer Product of Multivectors)**.**
The *Clifford outer product* of A_r and B_s, written with a *wedge* $\wedge_\circ$*,* is defined for any Clifford r-vector and s-vector $A_r, B_s \in \mathcal{C}\ell_{p,q}$ as the $(r+s)$-grade part of their Clifford product

$$A_r \wedge_\circ B_s := \langle A_r \circ B_s \rangle_{r+s}. \tag{5.2}$$

The Clifford outer product of general, not necessarily homogeneous, Clifford multivectors $A, B \in \mathcal{OC}_{p,q}$ is then defined by

$$A \wedge_\circ B := \sum_{r,s} \langle A \rangle_r \wedge_\circ \langle B \rangle_s = \sum_r \langle A \rangle_r \wedge_\circ B = \sum_s A \wedge_\circ \langle B \rangle_s. \tag{5.3}$$

Remark 5.6. In the above definition we have used the wedge $\wedge$, which is the usual symbol for the exterior product, with an added subscript $\circ$, which is the symbol for the Clifford product, to represent the Clifford outer product. Later, we also use similarly modified wedge, $\wedge_{\circledcirc}$, to represent the orientation congruent outer product. However, after presenting the last two axioms in Axiom Set IX, we have a bridge between the Clifford algebra and the orientation congruent algebra through a common outer product written with an unadorned wedge $\wedge$. For now, though, we use separate symbols for the Clifford and orientation congruent outer products.

Adding the seventh set of axioms turns the nonassociative algebra over $\mathbb{R}$, with a unit, a grade projection operator, r-blades, and r-vectors, into an associative algebra and relates the nondegenerate quadratic form Q associated with V^n to the Clifford square of the vectors in $\mathcal{C}\ell_{p,q}$. Our modified Hestenes-Sobczyk Clifford algebra fulfills all the requirements of Definition 5.30 taken from Lounesto, and is thus isomorphic to the Clifford algebra of the quadratic form $Q_{p,q}$ from generators and relations given by Definition 5.30.

Axioms VII.1 and VII.2 of Axiom Set VII have been placed at the end of our list of 26 axioms because these two will be substantially modified for the orientation congruent algebra $\mathcal{OC}_{p,q}$. We now continue and finish the Clifford algebra axiom set.

Axiom Set VII. *The $\mathcal{C}\ell_{p,q}$ Product: Specific Properties.*
For all Clifford algebras $\mathcal{C}\ell_{p,q}$, Clifford multiplication is such that for all $A, B, C \in \mathcal{C}\ell_{p,q}$

(VII.1) $(A \circ B) \circ C = A \circ (B \circ C)$. Associativity of the Clifford product[25]

For all Clifford algebras $\mathcal{C}\ell_{p,q}$, Clifford multiplication multiplication is such that for all $\mathbf{a} \in V^n$ there exists a function $Q^\circ_{p,q} \colon V^n \to \mathbb{R}$ satisfying Definitions 5.1 and 5.2 such that

(VII.2) $\mathbf{a} \circ \mathbf{a} = Q^\circ_{p,q}(\mathbf{a})$. Equality of the Clifford square of a vector and its quadratic form

Next, we list Theorem 5.7 for the Clifford algebra which is the direct counterpart of Axiom VII.1$'$ for the orientation congruent algebra.

[25]This Axiom VII.1, the associativity of Clifford multiplication, with the help of Axiom III.2, implies the (now redundant) Axiom II.3, the associativity of scalar multiplication.

Theorem 5.7 (Associativity of the Clifford Outer Product)**.** *For all Clifford algebras $\mathcal{C}\ell_{p,q}$, the Clifford outer product is such that for all $A, B, C \in \mathcal{C}\ell_{p,q}$*

$$(A \wedge_\circ B) \wedge_\circ C = A \wedge_\circ (B \wedge_\circ C). \tag{5.4}$$

Proof. The proof obviously involves Definition 5.5 and Axiom VII.1, but see Harke's paper [84, pp. 7, 11] or Hestenes and Sobczyk's book [97, pp. 7, 12] for the details. □

In preparation for Definition 5.9 we define the *exponential notation for the Clifford square* of any Clifford multivector by the following.

Definition 5.8 (Clifford Square Exponential Notation)**.**
For all multivectors $A \in \mathcal{C}\ell_{p,q}$

$$A^{\circ 2} := A \circ A. \tag{5.5}$$

The next definition extends to all vectors in the Clifford algebra $\mathcal{C}\ell_{p,q}$ the *sign and magnitude decomposition* of a scalar $\alpha \in \mathbb{R}$ given by the *signum* operator $\operatorname{sgn}(\bullet)$ and *absolute value* operator $|\bullet|$:

$$\alpha = \operatorname{sgn}(\alpha)\,|\alpha|.$$

Definition 5.9 (Clifford Sign and Magnitude of Vectors)**.**
Using Axiom VII.2 we may decompose any vector $\mathbf{a} \in V^n \subseteq \mathcal{OC}_{p,q}$ into two factors, a sign-determined normalized scalar, its *Clifford sign*[26] $\operatorname{sgn}_\circ(\mathbf{a}) \in \{-1, 0, 1\}$, and a scale-determined nonnegative scalar, its *Clifford magnitude* $|\mathbf{a}|_\circ \geq 0$, which are defined by the following equations. For all vectors $\mathbf{a} \in V^n$

$$\begin{aligned} &\operatorname{sgn}_\circ(\mathbf{a}) = |\mathbf{a}|_\circ = 0, && \text{if } \mathbf{a}^{\circ 2} = 0, \\ &\mathbf{a}^{\circ 2} = \operatorname{sgn}_\circ(\mathbf{a})\,|\mathbf{a}|_\circ^2, && \text{otherwise.} \end{aligned} \tag{5.6}$$

5.4. Modified HS Axioms for the Orientation Congruent Algebra $\mathcal{OC}_{p,q}$.

We consider now another list of 25 axioms parallel to the one above, but with the last two modified. Then, we add four new axioms to obtain a list of 29 axioms[27] that will provide the axiomatic foundation for the $\mathcal{OC}_{p,q}$ algebra.

Definitions 5.4, 5.5, 5.8, and the first 23 axioms in Axiom Sets I through VI are changed, but only trivially with the replacement of the terms and symbols referring to Clifford algebra with those referring to orientation congruent algebra. Therefore, we do not list Definitions 5.4, 5.5, 5.8, or the first 23 axioms in their modified forms.

The numbers of any *modified* definitions and axioms for the orientation congruent algebra, whether they are explicitly repeated in modified form or not, will be marked with primes to indicated their correspondence with the original axioms for the Clifford algebra. However, the numbers of the four new axioms, their axiom sets, and any new definitions will not be primed. Next, we briefly describe the material changes and additions to Axiom Set VII before making them.

[26] Hestenes, Li, and Rockwood in their book contribution [96, p. 2] use the name *signature* and symbol $\varepsilon_{\mathbf{a}}$ for this quantity, rather than our name *sign* and symbol $\operatorname{sgn}_\circ(\mathbf{a})$. However, our name and symbol are consistent with the analogous definition for real numbers. Thus, it fulfills the dictum that we should use uniform terms and symbols for multivectors of all grades.

[27] As with that for $\mathcal{C}\ell_{p,q}$ this axiom system for $\mathcal{OC}_{p,q}$ must also be supplemented with suitably modified conditions similar to (2) and (3) of Definition 5.30 and the requirement that $\mathbb{R}$ and V^n are distinct subspaces, again all adapted from Lounesto [123, p. 190].

The nontrivial changes to the Clifford algebra axioms in Axiom Set VII required to convert them to axioms for the orientation congruent algebra are

(1) restrict the product in Axiom VII.1, the associativity axiom, from the algebra product to the outer product,
(2) extend the domain of Axiom VII.2, the axiom for the equality of the algebra product square of a vector and its quadratic form to nonscalar blades, and
(3) add the two new Axioms VIII.1 and VIII.2, supplementing the now restricted Axiom VII.1′, that, in the given algebra, or, if necessary, the algebra derived from it by extending its base space one dimension higher, requires the existence of a *counit* $\boldsymbol{\omega}_{\mathscr{A}}$ *of a set of multivectors* $\mathscr{A}$ with the key property: *generalized* $\mathscr{A}$*-universal commutativity of the right* $\boldsymbol{\omega}_{\mathscr{A}}$*-complement.*

The first, explicitly modified axiom set uses Definition 5.5′ for the orientation congruent outer product of multivectors. Since this definition is obtained by trivially modifying the corresponding Clifford algebra Definition 5.5, we do not state it explicitly. However, recalling that the symbol for the Clifford outer product is $\wedge_\circ$, we do state that the analogous symbol for the orientation congruent outer product is $\wedge_{\circledcirc}$.

Axiom Set VII′ (modified)**.** *The* $\mathcal{OC}_{p,q}$ *Product: Specific Properties.*
For all orientation congruent algebras $\mathcal{OC}_{p,q}$, orientation congruent multiplication determines through Definition 5.5′ above the existence of the orientation congruent outer product as another algebraic product on the set $\mathcal{OC}_{p,q}$ such that for all $A, B, C \in \mathcal{OC}_{p,q}$

(VII.1′)	$(A \wedge_{\circledcirc} B) \wedge_{\circledcirc} C = A \wedge_{\circledcirc} (B \wedge_{\circledcirc} C).$	Associativity of the orientation congruent outer product

For all orientation congruent algebras $\mathcal{OC}_{p,q}$, orientation congruent multiplication is such that for any nonscalar r-blade $\mathbf{A} = \mathbf{a}_1 \circledcirc \cdots \circledcirc \mathbf{a}_i \circledcirc \cdots \circledcirc \mathbf{a}_r$ with pairwise anticommuting vectors $\mathbf{a}_i$, there exists a function $Q^{\circledcirc}_{p,q} \colon V^n \to \mathbb{R}$ satisfying Definitions 5.1 and 5.2 such that

(VII.2′)	$\mathbf{A} \circledcirc \mathbf{A} = Q^{\circledcirc}_{p,q}(\mathbf{a}_1) \cdots Q^{\circledcirc}_{p,q}(\mathbf{a}_i) \cdots Q^{\circledcirc}_{p,q}(\mathbf{a}_r).$	Equality of the $\mathcal{OC}$ square of an r-blade and the product of the quadratic forms of its vectors

The definition of the orientation congruent square notation, $A^{\circledcirc 2}$, is completely analogous to the definition of the Clifford square notation given in Definition 5.8. The next definition, however, similarly to Definition 5.9, extends to all blades in the orientation congruent algebra $\mathcal{OC}_{p,q}$ the *sign and magnitude decomposition* of a scalar $\alpha \in \mathbb{R}$ given by the *signum* $\operatorname{sgn}(\bullet)$ and *absolute value* $|\bullet|$ operators: $\alpha = \operatorname{sgn}(\alpha)\,|\alpha|$.

Definition 5.9′ (Orientation Congruent Sign and Magnitude of Blades)**.**
Using Axiom VII.2′ we may decompose any blade $\mathbf{A} \in \mathcal{OB}\ell_{p,q} \subseteq \mathcal{OC}_{p,q}$ into two factors, a sign-determined normalized scalar, its *orientation congruent sign* $\operatorname{sgn}_{\circledcirc}(\mathbf{A}) \in \{-1, 0, 1\}$, and a scale-determined nonnegative scalar, its *orientation congruent magnitude* $|\mathbf{A}|_{\circledcirc} \geq 0$, which are defined by the following equations. For

all blades $\mathbf{A} \in \mathcal{OB}\ell_{p,q}$

$$\begin{aligned} &\operatorname{sgn}_{\circledcirc}(\mathbf{A}) = |\mathbf{A}|_{\circledcirc} = 0, \quad \text{if } \mathbf{A}^{\circledcirc 2} = 0, \\ &\mathbf{A}^{\circledcirc 2} = \operatorname{sgn}_{\circledcirc}(\mathbf{A})\,|\mathbf{A}|_{\circledcirc}^{2}, \quad \text{otherwise.} \end{aligned} \tag{5.6$'$}$$

Now we may continue and finish with the last axiom set for the orientation congruent algebra comprising two new axioms. Axioms VIII.1 and VIII.2 define a *counit*. But first, we introduce a notation involving a counit that provides a naturally compact way to write the expressions in Axiom VIII.2 and the sequel.

Definition 5.10 (Counit Complementation Superscript Notation)**.**
Let $\mathscr{A} \subseteq \mathcal{OC}_{p,q}$ be any nonempty set of multivectors and $A \in \mathscr{A}$ be any multivector in $\mathscr{A}$. Then $\boldsymbol{\omega}_{\mathscr{A}}$ is a special multivector associated with $\mathscr{A}$, called a *counit of* $\mathscr{A}$, that we will defined in Axiom VIII.2. However, for now we only define the notation that will be used to state the axiom. We define a postfixed or prefixed superscript $\boldsymbol{\omega}_{\mathscr{A}}$ attached to A as

$$A^{\boldsymbol{\omega}_{\mathscr{A}}} := A \circledcirc \boldsymbol{\omega}_{\mathscr{A}}, \ \text{ or} \tag{5.7}$$

$$^{\boldsymbol{\omega}_{\mathscr{A}}}\!A := \boldsymbol{\omega}_{\mathscr{A}} \circledcirc A. \tag{5.8}$$

We call these operations *right* or *left* $\boldsymbol{\omega}_{\mathscr{A}}$*-complementation,* or *right* or *left counit complementation by* $\boldsymbol{\omega}_{\mathscr{A}}$*.* When written as superscripts we give these complementation operations precedence over all other operations, including orientation congruent, Clifford, and outer product multiplications.

The last definition we need to introduce before Axiom Set VIII$'$ follows.

Definition 5.11 (The Dimensional Extension[28] of $\mathcal{OC}_{p,q}$)**.**
Let $\mathcal{OC}_{p,q}$ be any orientation congruent algebra. Then the *dimensional extension* of $\mathcal{OC}_{p,q}$ means $\mathcal{OC}_{p,q}$ itself, or either of $\mathcal{OC}_{p+1,q}$ or $\mathcal{OC}_{p,q+1}$, the orientation congruent algebras (which always exist by Axiom VIII.1) derived from $\mathcal{OC}_{p,q}$ by arbitrarily extending its base space by one dimension. We write the dimensional extension of $\mathcal{OC}_{p,q}$ as $\mathcal{OC}_{(p,q)^+}$, its signature as $(p,q)^+$, and its dimension as n^+. We also say $\mathcal{OC}_{(p,q)^+}$ is a *dimensionally extended* orientation congruent algebra. We use the dimensional extension $\mathcal{OC}_{(p,q)^+}$, its signature, and its dimension as *metatheoretical* constructs, so that, for example, when we say $x \in \mathcal{OC}_{(p,q)^+}$ we mean $x \in \mathcal{OC}_{p,q}$, $x \in \mathcal{OC}_{p+1,q}$, or $x \in \mathcal{OC}_{p,q+1}$.

Axiom Set VIII (new)**.** *The* $\mathcal{OC}_{p,q}$ *Product: Counit Properties.*
Let $\mathcal{OC}_{p,q}$ be any orientation congruent algebra. Then

(VIII.1) $\quad \mathcal{OC}_{p+1,q}$ and $\mathcal{OC}_{p,q+1}$ exist, $\quad$ Existence of dimensional extension

and for all nonempty sets of multivectors $\mathscr{A} \subseteq \mathcal{OC}_{p,q}$, there exists a (nonunique) nonscalar, unit magnitude n^+-blade $\boldsymbol{\omega}_{\mathscr{A}} \in \mathcal{OC}_{(p,q)^+}$ with n^+ an odd integer, called a *counit* of $\mathscr{A}$, such that for all (not necessarily distinct) multivectors $A, B \in \mathscr{A}$

(VIII.2) $\quad A^{\boldsymbol{\omega}_{\mathscr{A}}} \circledcirc B = A \circledcirc B^{\boldsymbol{\omega}_{\mathscr{A}}} = (A \circledcirc B)^{\boldsymbol{\omega}_{\mathscr{A}}}.$ $\quad$ Generalized $\mathscr{A}$-universal commutativity of right $\boldsymbol{\omega}_{\mathscr{A}}$-complementation

[28] The development of this formulation was initially prompted by John Browne's [32] suggestion that even-dimensional base spaces be included in the definition of an orientation congruent algebra.

Theorem 5.12 ($\mathscr{A}$-Universal Commutativity of $\boldsymbol{\omega}_{\mathscr{A}}$)**.** *For all nonempty sets of multivectors $\mathscr{A} \subseteq \mathcal{OC}_{p,q}$ within any orientation congruent algebra $\mathcal{OC}_{p,q}$, there exists a counit $\boldsymbol{\omega}_{\mathscr{A}} \in \mathcal{OC}_{(p,q)^+}$, such that for all multivectors $A \in \mathscr{A}$*

$$A \circledcirc \boldsymbol{\omega}_{\mathscr{A}} = \boldsymbol{\omega}_{\mathscr{A}} \circledcirc A. \tag{5.9}$$

Proof. The proof follows immediately from Definition 5.10 and Axiom VIII.2 by setting $B = 1$. □

Precisely now with the presentation of the final axiom in Axiom Set VIII′ we have completed the construction a modified Hestenes-Sobczyk axiom system for the orientation congruent algebra of a nondegenerate quadratic form. However, because of the pivotal role of the counits in the orientation congruent algebra we are about to enter into a long series of definitions and remarks related to them.

These definitions and remarks will be followed by one more, final axiom, that is not an axiom of either algebra alone, but that acts as a bridge between the Clifford and orientation congruent algebras of the same vector space and quadratic form. This final axiom allows the two algebras to be knit together into a one system with two fundamental products—the Clifford product and the orientation congruent product—as well as one common outer product.

Remark 5.13. We normally use the single word *counit* which is a contraction of the phrase *coscalar unit*. However, to avoid confusion when working with Hopf or similar algebraic theories which use the term *counit* for an unrelated concept, we may employ the full phrase *coscalar unit*. The "unit" part of this name is appropriate because a counit behaves algebraically like the unit. Indeed, for the set $\mathscr{A} = \mathcal{OC}_{p,q}$ 1 and -1 are the only elements other than a counit of $\mathcal{OC}_{p,q}$ or its negative, $\pm\boldsymbol{\omega}_{\mathscr{A}}$, that are of unit magnitude and satisfy Axiom VIII.2 (excepting that they are scalars). Also, the "co" part of the name is consistent with the definition of a *coscalar* as an element of $\mathcal{OC}_{p,q}$ that has a *complementary grade* or *cograde* of $0 = n^+ - k$ because it has a grade of $k = n^+$ in the set of multivectors $\mathcal{OC}_{(p,q)^+}$ with $n^+ = p + q$ or $n^+ = p + q + 1$. Generally, when working in the algebra $\mathcal{OC}_{p,q}$, a minimal grade counit $\boldsymbol{\omega}_{\mathscr{A}}$ of a nonempty set of multivectors $\mathscr{A}$ has a cograde of $0 = m - k$ (or a grade of $k = m$) *relative to* the smallest odd $m = r + s \leq n^+$ such that $\mathscr{A} \subseteq \mathcal{OC}_{r,s} \subseteq \mathcal{OC}_{(p,q)^+}$.

Next, applying the last axiom, we make two sets of definitions. The first set characterizes the counits defined by Axiom VIII.2 as being *intrinsic, extrinsic*, or *nonintrinsic* to the orientation congruent algebra $\mathcal{OC}_{p,q}$. The second set extends the usage of the word *counit* to the unit-magnitude, maximal-grade blades in $\mathcal{OC}_{p,q}$ by qualifying it with the three adjectives *perfect, imperfect*, or *indefinite*. Later, in the heart of the paper, the terminology of this second definition will prove to be very convenient.

Definition 5.14 (Algebra-Intrinsic, -Extrinsic, and -Nonintrinsic Counits)**.**

(1) If the there exists a counit $\boldsymbol{\omega}_{\mathscr{A}}$ of Axiom VIII.2 or Theorem 5.12 such that it is not only an element of the dimensionally extended orientation congruent algebra $\mathcal{OC}_{(p,q)^+}$, but also an element of the given orientation congruent algebra $\mathcal{OC}_{p,q}$, we say that it is an *algebra-intrinsic* counit of $\mathscr{A}$. The counit of $\mathcal{OC}_{p,q}$ is algebra-intrinsic if, and only if, $n = p + q$ is an odd integer.

(2) If that counit $\boldsymbol{\omega}_{\mathscr{A}}$ is not an element of $\mathcal{OC}_{p,q}$ we say that it is an *algebra-extrinsic* counit of $\mathscr{A}$.

(3) In any case, without further information, we may say that the counit of Axiom VIII.2 or Theorem 5.12 is an *algebra-nonintrinsic* counit, meaning a *not necessarily algebra-intrinsic* counit.

Definition 5.15 (Perfect, Imperfect, and Indefinite Counits and $\mathcal{OC}$ Algebras)**.**

(1) If $\mathscr{A} = \mathcal{OC}_{p,q}$, we may call any algebra-intrinsic counit $\boldsymbol{\omega}_{\mathscr{A}}$ of $\mathcal{OC}_{p,q}$ a *perfect counit of the algebra* $\mathcal{OC}_{p,q}$. We usually write such a counit with a boldface uppercase omega as $\boldsymbol{\Omega}$. If any orientation congruent algebra $\mathcal{OC}_{p,q}$ has a perfect counit, we may call that algebra a *perfect* orientation congruent ($\mathcal{POC}$) algebra and write it as $\mathcal{POC}_{p,q}$.

(2) If $\mathscr{A} = \mathcal{OC}_{p,q}$ and $\mathcal{OC}_{p,q}$ does not have an algebra-intrinsic counit $\boldsymbol{\omega}_{\mathscr{A}}$, we may call any unit-magnitude n-blade of $\mathcal{OC}_{p,q}$ an *imperfect counit of the algebra* $\mathcal{OC}_{p,q}$. We usually write such a counit with an upside-down boldface uppercase omega as $\boldsymbol{\mho}$. If any orientation congruent algebra $\mathcal{OC}_{p,q}$ does not have a perfect counit, we may call that algebra an *imperfect* orientation congruent ($\mathcal{IOC}$) algebra and write it as $\mathcal{IOC}_{p,q}$.

(3) In any case, without further information about whether the orientation congruent algebra $\mathcal{OC}_{p,q}$ has an algebra-intrinsic counit or not, we may call any unit-magnitude n-blade of $\mathcal{OC}_{p,q}$ an *indefinite counit of the algebra* $\mathcal{OC}_{p,q}$. We usually write such a counit with the symbol $\rlap{\boldsymbol{\Omega}}\boldsymbol{\mho}$. This symbol, $\rlap{\boldsymbol{\Omega}}\boldsymbol{\mho}$, was designed to resemble the superposition of a handwritten boldface uppercase omega $\boldsymbol{\Omega}$ and a handwritten upside-down boldface uppercase omega $\boldsymbol{\mho}$. When we do not know, or do not wish to specify, whether an orientation congruent algebra $\mathcal{OC}_{p,q}$ has an algebra-intrinsic counit or not, we call that algebra simply an *orientation congruent ($\mathcal{OC}$) algebra* and write it as $\mathcal{OC}_{p,q}$, just as we have done until now.

Remark 5.16. As we have postulated in Axiom VIII.1, the imperfect orientation $\mathcal{IOC}_{p,q}$ with even $n = p + q$ can always be extended by one dimension to the perfect orientation congruent algebra $\mathcal{POC}_{p',q'}$ with odd $n' = p' + q' = n + 1$. We can create this perfect orientation congruent algebra from $\mathcal{IOC}_{p,q}$, by letting $\mathcal{POC}_{p',q'}$ have the set of primed basis vectors $\mathscr{B}' = \{\mathbf{e}'_1, \ldots \mathbf{e}'_{n+1}\}$ obtained by adding one more basis vector to some signature-ordered, orthogonal set of basis vectors $\mathscr{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_p, \mathbf{e}_{p+1}, \ldots, \mathbf{e}_n\}$ for $\mathcal{IOC}_{p,q}$. The new basis vector must be orthogonal to the original ones, but it could be either $\mathbf{e}'_{p'} = \mathbf{e}'_{p+1}$ with $Q\left(\mathbf{e}'_{p'}\right) > 0$, which makes $p' = p + 1$ and $q' = q$), or $\mathbf{e}'_{n'} = \mathbf{e}'_{n+1}$ with $Q\left(\mathbf{e}'_{n'}\right) < 0$, which makes $p' = p$ and $q' = q+1$. The imperfect orientation congruent algebra $\mathcal{IOC}_{p,q}$ is actually a *subalgebra* of either of $\mathcal{POC}_{p+1,q}$ or $\mathcal{POC}_{p,q+1}$, the two next higher-dimensional perfect orientation congruent algebras with compatible signatures.

Remark 5.17. The q part of the signature (p, q) of the quadratic form associated with $\mathcal{OC}_{p,q}$ determines the sign of the orientation congruent square of an indefinite counit of the algebra by $\operatorname{sgn}(\rlap{\boldsymbol{\Omega}}\boldsymbol{\mho}) = \rlap{\boldsymbol{\Omega}}\boldsymbol{\mho}^{\circledcirc 2} = (-1)^q$.

We now extend the superscript notation for counit complementation to these new counit concepts and symbols with the following definition. This notation will prove its worth in a later section of the paper.

Definition 5.18 (Extended Counit Complementation Superscript Notation)**.**
Let $\mathbf{O}_n$ be any of types of extended counits of the orientation congruent algebra

$\mathcal{OC}_{p,q}$ specified in Definition 5.15, namely, a perfect counit, $\boldsymbol{\Omega}$, an imperfect counit, $\boldsymbol{\mho}$, or an indefinite counit, $\overline{\boldsymbol{\Omega}}$. Also, let $A \in \mathcal{OC}_{p,q}$ be any general multivector in $\mathcal{OC}_{p,q}$. Then we define a postfixed or prefixed superscript $\mathbf{O}_n$ attached to A as

$$A^{\mathbf{O}_n} := A \circledcirc \mathbf{O}_n, \text{ or}$$
$${}^{\mathbf{O}_n}A := \mathbf{O}_n \circledcirc A.$$

We call these operations *right* or *left, perfect (imperfect, indefinite) counit complementation*, or *right* or *left complementation by a perfect (imperfect, indefinite) counit.* When written as superscripts we give them precedence over all other operations, including orientation congruent, Clifford, and outer product multiplications.

Next, we extend this superscript notation for counit complementation to the unit-magnitude pseudoscalars of a Clifford algebra with the following definition involving Clifford multiplication rather than orientation congruent multiplication. This notation will also be useful in a later section of the paper.

Definition 5.19 (Pseudoscalar Complementation Superscript Notation)**.**
In the applied geometric (Clifford) algebra literature the term *pseudoscalar* is used for a general, maximum grade blade in a geometric algebra with a base space V^n of fixed dimension n. Now let $\mathbf{I}$ be a *unit-magnitude* pseudoscalar of the Clifford algebra $\mathcal{C}\ell_{p,q}$. Also let $A \in \mathcal{C}\ell_{p,q}$ be any general multivector in $\mathcal{C}\ell_{p,q}$. Then we define a prefixed or postfixed superscript $\mathbf{I}$ attached to A as

$${}^{\mathbf{I}}A := \mathbf{I} \circ A, \text{ and}$$
$$A^{\mathbf{I}} := A \circ \mathbf{I}.$$

We call these operations, respectively, *left* and *right pseudoscalar complementation*, or *left* and *right complementation by a pseudoscalar*. When written as superscripts we give them precedence over all other operations, including orientation congruent, Clifford, and outer product multiplications.

Finally, we define a pseudoscalar with a particular orientation as *the* pseudoscalar and a counit with a particular orientation as *the* counit.

Definition 5.20 (*The* pseudoscalar and *the* counit)**.**

(1) For the Clifford algebra $\mathcal{C}\ell_{p,q}$, there are exactly two oppositely-oriented, unit-magnitude pseudoscalars, $\mathbf{I}$ and $-\mathbf{I}$, that differ only by sign. Often in the applied geometric algebra literature and always in this paper, the phrase ***the*** *pseudoscalar* is used to refer to a unit-magnitude pseudoscalar $\mathbf{I}$ with a definite, explicitly assigned, orientation. If an explicit orientation for $\mathbf{I}$ is not mentioned, but an ordered, orthonormal set of basis vectors for V^n, $\mathscr{B} = \{\, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \,\}$, is available, we assume that $\mathbf{I}$ is assigned the orientation that is compatible with that of V^n, namely, $\pm \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n$.

(2) Similarly, for the orientation congruent algebra $\mathcal{OC}_{p,q}$, there are exactly two oppositely-oriented, indefinite counits, $\overline{\boldsymbol{\Omega}}$ and $-\overline{\boldsymbol{\Omega}}$, that differ only by sign. The phrase ***the*** *indefinite counit of the algebra* $\mathcal{OC}_{p,q}$, or, simply, ***the*** *indefinite counit* is used to refer to an indefinite counit with a definite, explicitly assigned, orientation. To distinguish it from its oppositely oriented companion we may sometimes write the indefinite counit with an underline as $\underline{\overline{\boldsymbol{\Omega}}}$. If an explicit orientation for $\overline{\boldsymbol{\Omega}}$ is not mentioned, but an ordered, orthonormal set of basis vectors for V^n, $\mathscr{B} = \{\, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \,\}$, is available,

we assume that $\boldsymbol{\Omega\mkern-13mu\mho}$ is assigned the orientation that is compatible with that of V^n, namely, $\pm\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n$.

(3) We extend the terms and notations of this definition for the indefinite counit of the orientation congruent algebra $\mathcal{OC}_{p,q}$ to the perfect counit $\boldsymbol{\Omega}$ of the perfect orientation congruent algebra $\mathcal{POC}_{p,q}$ and the imperfect counit $\boldsymbol{\mho}$ of the imperfect orientation congruent algebra $\mathcal{IOC}_{p,q}$.

Remark 5.21. Usually we work with one ordered, orthonormal set of basis vectors, $\mathscr{B} = \{\, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \,\}$, shared by both the Clifford algebra $\mathcal{C}\ell_{p,q}$ and the orientation congruent algebra $\mathcal{OC}_{p,q}$. However, according to the conventions of Definitions 5.18 and 5.19, for an indefinite counit $\boldsymbol{\Omega\mkern-13mu\mho}$ that has the same orientation as a pseudoscalar $\mathbf{I}$, it turns out that although $\boldsymbol{\Omega\mkern-13mu\mho}$ is equal to $\mathbf{I}$, always, $A^{\boldsymbol{\Omega\mkern-13mu\mho}}$ is not necessarily equal to $A^{\mathbf{I}}$, and neither is ${}^{\boldsymbol{\Omega\mkern-13mu\mho}}\!A$ necessarily equal to ${}^{\mathbf{I}}\!A$, for a general multivector A and a general dimension n:

$$A^{\boldsymbol{\Omega\mkern-13mu\mho}} \neq A^{\mathbf{I}} \quad \text{in general, and}$$
$${}^{\boldsymbol{\Omega\mkern-13mu\mho}}\!A \neq {}^{\mathbf{I}}\!A \quad \text{in general.}$$

Remark 5.22.

(1) The penultimate axiom, Axiom VIII.2, along with Axiom VII.1′ replaces Axiom VII.1 expressing the associativity of the Clifford product. Associativity is just one (the simplest) member of the class of possible bracket shifting rules.

(2) Axiom VII.1′ partially replaces the general associativity of the Clifford product with that of the orientation congruent outer product. The orientation congruent outer product is derived by a grade selection from the orientation congruent product. Equivalently, this axiom may be restated to postulate the associativity of the orientation congruent product of the component vectors of two blades if those component vectors mutually anticommute under the orientation congruent product when combined as one group. This axiom has a direct analog as a theorem in all Clifford algebras $\mathcal{C}\ell_{p,q}$.

(3) Axiom VIII.2 supplements Axiom VII.1′ with a bracket shifting rule involving $\boldsymbol{\omega}_{\mathscr{A}}$, and is more complicated, but generally applicable. Axiom VII.1′ has a direct analog as a theorem in all Clifford algebras. But Axiom VIII.2 has a direct analog as a theorem in only Clifford algebras $\mathcal{C}\ell_{p,q}$ with odd $n = p + q$.

(4) In summary, we might say that to transform the axioms for $\mathcal{C}\ell_{p,q}$ into those for $\mathcal{OC}_{p,q}$ we have traded an expansion of the domain of applicability of Axiom VII.2 from vectors to blades in Axiom VII.2′ for a restriction of the domain of applicability of Axiom VII.1 with its consequent fragmentation into the two Axioms VII.1′ and VIII.2.

5.5. **Algebra Fusion.** Now, after having developed the Clifford and orientation congruent algebras separately we present a final, bridge axiom set. Its two axioms, properly speaking, belong to neither the Clifford algebra, $\mathcal{C}\ell_{p,q}$ nor the orientation congruent algebra, $\mathcal{OC}_{p,q}$. However, we can use the axioms of Axiom Set IX to knit the Clifford and orientation congruent algebras together into a kind of generalized Kähler-Atiyah algebra which contains the Clifford, orientation congruent, and outer (exterior) products, as well as various flavors of Clifford and orientation congruent inner products and contraction operators.

Axiom Set IX (new)**.** *Clifford and Orientation Congruent Algebra Bridge.* Let $\mathcal{C}\ell_{p,q}$ be any Clifford algebra and let $\mathcal{OC}_{p',q'}$ be any orientation congruent algebra such that $p = p'$, $q = q'$. Then there is an isomorphism $f\colon V_\circ^{p,q} = \mathcal{C}\ell^1_{p,q} \to V_\circledcirc^{p',q'} = \mathcal{OC}^1_{p',q'}$ from the Clifford to the orientation congruent product of vectors which satisfies the following equations:

(IX.1)

$$\mathbf{x} \circ \mathbf{y} + \mathbf{y} \circ \mathbf{x} = f(\mathbf{x}) \circledcirc f(\mathbf{y}) + f(\mathbf{y}) \circledcirc f(\mathbf{x}),$$

Isomorphism of the Clifford and orientation congruent anticommutator for vectors

(IX.2)

$$\mathbf{x} \circ \mathbf{y} - \mathbf{y} \circ \mathbf{x} = f(\mathbf{x}) \circledcirc f(\mathbf{y}) - f(\mathbf{y}) \circledcirc f(\mathbf{x}).$$

Isomorphism of the Clifford and orientation congruent commutator for vectors

After adding the two axioms in Axiom Set IX the results cataloged in the following remarks follow. However, we leave all proofs to the reader, reminding that the sources [84] [97], [123], and [124] are available for some.

Remark 5.23. Henceforth, we identify isomorphic vectors from each algebra as well as the two quadratic forms. Therefore, we may use a common notation for the vectors spaces and quadratic forms and simply write $V^{p,q} := \mathcal{C}\ell^1_{p,q} = \mathcal{OC}^1_{p,q}$ and $Q_{p,q} := Q^\circ_{p,q} = Q^\circledcirc_{p,q}$. Then, for all $\mathbf{x}, \mathbf{y} \in V^{p,q}$, we have

$$\mathbf{x} \circ \mathbf{y} = \mathbf{x} \circledcirc \mathbf{y}.$$

Remark 5.24. Any vector $\mathbf{x} \in V^{p,q}$ has a common square in the two algebras, which we may write simply as $\mathbf{x}^2$:

$$\mathbf{x}^2 := \mathbf{x}^{\circ 2} = \mathbf{x}^{\circledcirc 2}.$$

Remark 5.25. For all sets of r vectors, that pairwise anticommute in either algebra, their Clifford and orientation congruent *outer* multiproducts are equal to each other and to their Clifford and orientation congruent multiproducts. Therefore we may simply write all multiproducts of these vectors with the same symbol $\wedge$:

$$\begin{aligned} x_1 \wedge \cdots \wedge x_i \wedge \ldots \wedge x_r &= x_1 \wedge_\circ \cdots \wedge_\circ x_i \wedge_\circ \ldots \wedge_\circ x_r \\ &= x_1 \wedge_\circledcirc \cdots \wedge_\circledcirc x_i \wedge_\circledcirc \ldots \wedge_\circledcirc x_r \\ &= x_1 \circ \cdots \circ x_i \circ \ldots \circ x_r \\ &= x_1 \circledcirc \cdots \circledcirc x_i \circledcirc \ldots \circledcirc x_r. \end{aligned}$$

Thus, all Clifford multivectors are equal to their orientation congruent counterparts, and we have one set of multivectors, which we write as $\mathcal{CO}_{p,q}$, with three products $\circ$, $\circledcirc$, and $\wedge$. We name the algebra of this set and its three products the *Clifford-orientation-congruent* algebra.

Remark 5.26. We agree to use the common symbols: $\mathcal{B}\ell$ for the set of blades, $\mathscr{B}$ for an orthonormal set of basis vectors, and $\mathcal{B}\ell_{\mathscr{B}}$ for a (nonunique) set of basis blades derived from $\mathscr{B}$. We extend by analogy the use of all symbols previously defined separately for the Clifford and orientation congruent algebras to the common algebra $\mathcal{CO}_{p,q}$. Specifically according the above remarks, we have the following equalities of sets and, generally, in the sequel we prefer to use the first symbol in each line:

$$\begin{aligned}
\mathcal{B}\ell_{\mathscr{B}} &:= \mathcal{CB}\ell_{\mathscr{B}} = \mathcal{OB}\ell_{\mathscr{B}}, && \text{Basis } r\text{-blades}\\
\mathcal{B}\ell_{\mathscr{B}} &:= \mathcal{CB}\ell_{\mathscr{B}} = \mathcal{OB}\ell_{\mathscr{B}}, && \text{Basis blades}\\
\mathcal{B}\ell^r_{p,q} &:= \mathcal{CB}\ell^r_{p,q} = \mathcal{OB}\ell^r_{p,q}, && r\text{-Blades}\\
\mathcal{B}\ell_{p,q} &:= \mathcal{CB}\ell_{p,q} = \mathcal{OB}\ell_{p,q}, && \text{Blades}\\
\mathcal{CO}^r_{p,q} &:= \mathcal{C}\ell^r_{p,q} = \mathcal{OC}^r_{p,q}, && r\text{-Vectors}\\
\mathcal{CO}_{p,q} &:= \mathcal{C}\ell_{p,q} = \mathcal{OC}_{p,q}. && \text{Multivectors}
\end{aligned}$$

With the help of Axioms IX.1 and IX.2 we can establish a more general compatibility relation between the Clifford and orientation congruent products of any two blades that jointly anticommute. However, we must first define the meaning of the phrase *compatible blades*.

Definition 5.27 (Compatible Blades)**.**
We say that any two nonscalar blades $\mathbf{A}, \mathbf{B} \in \mathcal{B}\ell$ are *compatible* if and only if there exists a set of vectors $\mathcal{C}$ that pairwise anticommute and both $\mathbf{A}$ and $\mathbf{B}$ can be expressed as a product of vectors from $\mathcal{C}$ and a scalar factor. By extension, we also define any scalar and to be compatible with any blade.

Theorem 5.28 ($\mathcal{C}\ell$ and $\mathcal{OC}$ Product Compatibility of Compatible Blades)**.** *For all blades $\mathbf{A}, \mathbf{B} \in \mathcal{B}\ell$ that are compatible there exists a scalar $\alpha \in \{1, -1\}$ such that*

$$\mathbf{A} \circ \mathbf{B} = \alpha \mathbf{B} \circ \mathbf{A} \tag{5.9}$$

if and only if

$$\mathbf{A} \circledcirc \mathbf{B} = \alpha \mathbf{B} \circledcirc \mathbf{A}. \tag{5.10}$$

Proof. The proof is left to the reader. □

Remark 5.29. In a later section, after adding to the Clifford, orientation congruent, and outer products various derived inner products and contraction operators, we see that the Clifford-orientation-congruent algebra is a kind of generalized Kähler-Atiyah algebra. These derived inner products and contraction operators are different depending on whether they arise from the Clifford or orientation congruent products.

5.6. **Other Axiom Systems.** Before we give a set of axioms for $\mathcal{OC}_{p,q}$ we first introduce a compact axiomatic definition of $\mathcal{C}\ell_{p,q}$ adapted from Lounesto's presentation.[29] Then we will expand this compact definition into a longer list of 25 axioms in seven sets. Finally, after modifying this axiomatic formulation for $\mathcal{C}\ell_{p,q}$, we obtain a system of 27 axioms for $\mathcal{OC}_{p,q}$.

Hereafter the term *multivector* shall refer to any element of the Clifford algebra $\mathcal{C}\ell_{p,q}$ (or the orientation congruent algebra $\mathcal{OC}_{p,q}$) including those containing a scalar or vector component. Also the Clifford algebra product shall be denoted by an open dot $\circ$.[30]

[29] See Lounesto's book [123, pp. 190–192]. Chapters 14, 21, and 22 of the same book [123] also give several other definitions of a Clifford algebra.

[30] Usually Clifford multiplication is indicated by juxtaposition but here we prefer to distinguish between it and orientation congruent multiplication by giving each its own symbol: an open dot $\circ$, and a circled open dot $\circledcirc$, respectively.

Definition 5.30 (The Clifford Algebra $\mathcal{C}\ell_{p,q}$ Defined by Generators & Relations)**.**

This definition is taken from Lounesto's publications [123, p. 190], [124, pp. 26–27]. An associative algebra over $\mathbb{R}$ with unit 1 is the *Clifford algebra* $\mathcal{C}\ell_{p,q}$ of a nondegenerate quadratic form Q on V^n (with the Clifford product symbolized by an open dot $\circ$) if it contains V^n and $\mathbb{R} = \mathbb{R} \cdot 1$ as distinct subspaces so that

(1) $x \circ x = Q(x)$ for all $x \in V^n$,
(2) V^n generates $\mathcal{C}\ell_{p,q}$ as an algebra over $\mathbb{R}$, and
(3) $\mathcal{C}\ell_{p,q}$ is not generated by any proper subset of V^n.

As Lounesto remarks condition (3) of Definition 5.30 ensures that $\mathcal{C}\ell_{p,q}$ so defined is a *universal object* in the category theoretic sense and that the dimension of $\mathcal{C}\ell_{p,q}$ is 2^n. Roughly stated, the universality of a Clifford algebra means that it is unique up to isomorphism under a change of orthonormal basis and that it is of the maximum size allowed by its definition.[31] Applied works commonly use a long set of axioms similar to those we give next to define the Clifford algebra $\mathcal{C}\ell_{p,q}$; however, usually their authors do not also mention the refinement of condition (3).

The literature provides other axiomatic formulations of Clifford algebras of varying generality. Here we will consider their adaptability to the orientation congruent algebra.[32]

These other Clifford algebra axiom systems range, for example, from those describing a Clifford algebra as an ideal of a tensor algebra [123, pp. 193 f.], or describing it in category-theoretic terms as the universal object of a quadratic algebra [123, pp. 192 f.], or embedding it as a subalgebra of the associated exterior algebra's endomorphism algebra through the Chevalley-operator representation (which Chevalley [44] based on the Cartan decomposition formula),[33] or describing it as a Hopf gebra[34] using tensor algebra and category theory expressed in commutative and tangle diagrams [68, Chs. 3–5], to providing a multiplication rule for basis blades represented by n-tuples of binary digits called multi-indices ([123], ch. 21).[35]

Only three of these approaches to the axiomatization of Clifford algebra are directly convertible to the orientation congruent algebra. One is the definition as a universal object of quadratic algebras. The modification required is simply using nonassociative quadratic algebras in place of the (assumed) associative quadratic algebras and adding other relations to represent Axioms VII.1′, VII.2′, and VIII.2. However, since this very abstract definition is nonconstructive, it is not useful for calculating the orientation congruent product.

[31] For a more detailed discussion of universality under the name *unique factorization property,* and in the context of the tensor product of vector spaces, see Shaw [167, pp. 274–277]. For a specifically Clifford algebraic discussion see Gilbert and Murray's book [77, pp. 12–17] or the brief treatment in Perwass's thesis [139, p. 18]. For the related category theoretic formulation of Clifford algebras see Lounesto's book chapter [124, pp. 26–29]. Lastly, for German readers, the discussion in Florian's thesis [109, app. A.4] appears to be good.

[32] Subsection 8.1 has more remarks on axiomatizations.

[33] This decomposition formula is credited to E. Cartan by Crumeyrolle [51, p. 44] and Abłamowicz [2, p. 463]. Chevalley's method is also used by Lounesto [123, ch. 22], Crumeyrolle [51, p. 45], and Oziewicz [135]. It is also implicit in the paper of Fernández, Moya, and Rodrigues [69, p. 15]. Also see Subsection 8.1 for more remarks on axiomatizations.

[34] This is not a misprint. Without going into details, a Hopf gebra is a more general structure than a Hopf algebra [68, p. 65].

[35] This last is really a specialized form of GR axiomatization.

It is only the last two definitions, one based on Hopf gebra and the other on a multiplication rule for basis blades that are both adaptable and useful. That is because the other approaches are based on intrinsically associative algebras. Hopf gebras, however, are not ruled out; associativity is not necessary for their definition [68, p. 65]. Also as demonstrated by Fauser [68] the Hopf gebraic approach is very fruitful in producing grade-free computational algorithms for very general forms of Clifford algebras.

The last definition from a multiplication rule for basis blades is easily generalizable to Clifford-like algebras. These are essentially the algebras of the Clifford product but as modified by a sign rule that may differ from the standard Clifford algebra one [123, pp. 284 ff.]. The Clifford-like algebras, however, are not necessarily associative. They may also have other properties that vary from those of the Clifford algebras. In the following section we will construct the explicitly Clifford-like *sigma orientation congruent algebra* $\sigma\mathcal{OC}_{p,q}$. As suggested above we will fashion the product of the sigma orientation congruent algebra from the Clifford product times a *sign factor function* σ.

In Section 6 we also prove the deductive equivalence of the set of primed axioms for the orientation congruent algebra $\mathcal{OC}_{p,q}$ with that of the unprimed axioms for the Clifford algebra $\mathcal{C}\ell_{p,q}$ supplemented by an existence axiom for the sigma orientation congruent product. In so doing we establish that the sigma orientation congruent algebra of a nondegenerate quadratic form is isomorphic to the corresponding orientation congruent algebra. Then, instead of reasoning directly from the axioms of the current section, we can also prove theorems for the orientation congruent algebra by interpreting its product as the sigma orientation congruent product and manipulating ordinary algebraic expressions derived from the sign factor function while citing verified Clifford algebra theorems.

Actually, in the sequel to Section 6 the sigma form of the orientation congruent product will be the basis for investigating the $\mathcal{OC}_{p,q}$ algebra. Indeed, in Section 6 while simply proving the equivalence of the and the sigma orientation congruent product other proofs of some assertions made in this section will naturally fall out as byproducts. One statement with such an incidental proof is that a perfect orientation congruent algebra $\mathcal{POC}_{p,q}$ exists in all and only those base spaces V^n of odd dimension, or, complementarily, that an imperfect orientation congruent algebra $\mathcal{IOC}_{p,q}$ exists in all and only those base spaces V^n of even dimension.

5.7. **Derivation of the $\mathcal{OC}_3$ Multiplication Table.** In this subsection we derive the multiplication table for the orientation congruent algebra $\mathcal{OC}_3$ from the modified Hestenes-Sobczyk axioms of Subsection 5.4. Let us consider some of the conventions used here.

The multiplication table for $\mathcal{OC}_3$ is expressed in terms of factors that are all the positively-signed basis blades formed from an orthonormal set basis of basis vectors for the base space V^n. In the tables of this subsection and the next the basis blades are written with multi-indices so that, for example, $\mathbf{e}_{23} = \mathbf{e}_2 \circ \mathbf{e}_3$ or $\mathbf{e}_2 \circledcirc \mathbf{e}_3$ depending on which algebra appears in the table. However, as we show in a later section of this paper, for basis blades derived from an orthonormal set of basis vectors, these two expressions may be identified and equated with $\mathbf{e}_2 \wedge \mathbf{e}_3$, an outer product common to both the Clifford and the orientation congruent algebras.

In addition, we use a particular standard set of basis blades in these tables. The multi-indices of these standard basis blades are ordered in sequences that are natural for the orientation congruent algebra. Thus we write $\mathbf{e}_{31}$ and not $\mathbf{e}_{13}$.

Partial results for the derivation of multiplication table of $\mathcal{OC}_3$ are presented in Tables 5.1 and 5.2 with the scalar constants α, β, and γ substituted for the values to be determined. The final multiplication table for $\mathcal{OC}_3$ is presented as Table 5.6 of the next subsection. Also in the next subsection we give the multiplication tables of more orientation congruent algebras and, for comparison, some of the Clifford algebras with the same base space and quadratic form.

The red tinted cells of Table 5.1 contain all the products that can easily be determined by applying the axioms in Axioms Sets I′ through VII′. The untinted cells of Table 5.1 contain the subscripted constants α, β, and γ. The values of these constants can be only 1 or -1. The subscripting scheme used for the α, β, and γ constants will be explained below in the course of determining their values.

First, using the axioms of Axiom Set VIII, we settle the value of the γ's which are associated with products which have the perfect counit $\mathbf{\Omega}$ as one factor. The subscript of a γ for a given product is assigned to be the same as the multi-index of the basis blade multiplying $\mathbf{\Omega}$ in that product.

Using Axiom VIII.2, here are the calculations for determining γ_1:

$$
\begin{aligned}
&(\mathbf{e}_1 \circledcirc \mathbf{e}_1) \circledcirc \mathbf{\Omega} = \mathbf{\Omega} \\
&\mathbf{e}_1 \circledcirc (\mathbf{e}_1 \circledcirc \mathbf{\Omega}) = \mathbf{e}_1 \circledcirc \gamma_1 \mathbf{e}_{23} = \gamma_1 \mathbf{\Omega}.
\end{aligned}
$$

The result, $\gamma_1 = 1$, follows from Axiom VIII.2 by equating the very last expressions in the two lines of equations above. By following this pattern *mutatis mutandis* the γ's associated with the remaining basis blades may also be determined to be 1. These values have been entered into Table 5.2.

Next we derive values for the α constants. These constants are associated with products between vector and bivector basis blades. The α's are indexed according to the following scheme. The modified multi-index of an α associated with a given product is the same as that of the bivector factor involved in the product. However, the integers in the α's modified multi-index are not necessarily in the same order as they are in the bivector factor. The order of the integers in an α multi-index is such that same integer as that of the vector factor involved in the product is placed in the same position of the α's multi-index as the vector factor itself occupies in the product. Finally, to distinguish that integer from the other one in an α's multi-index, a bar is placed over the integer that is the same as the vector factor's index.

Consider now the following products which follow the pattern of Axiom VIII.2:

$$
\begin{aligned}
&(\mathbf{e}_1 \circledcirc \mathbf{\Omega}) \circledcirc \mathbf{e}_2 = \mathbf{e}_{23} \circledcirc \mathbf{e}_2 = \alpha_{3\bar{2}} \mathbf{e}_3 \\
&\mathbf{e}_1 \circledcirc (\mathbf{e}_2 \circledcirc \mathbf{\Omega}) = \mathbf{e}_1 \circledcirc \mathbf{e}_{31} = \alpha_{\bar{1}3} \mathbf{e}_3 \\
&(\mathbf{e}_1 \circledcirc \mathbf{e}_2) \circledcirc \mathbf{\Omega} = \mathbf{e}_{12} \circledcirc \mathbf{\Omega} = \mathbf{e}_3.
\end{aligned} \tag{5.11}
$$

Commuting $\mathbf{e}_1$ and $\mathbf{e}_2$ in the above set of equations leads to the following set:

$$
\begin{aligned}
&(\mathbf{e}_2 \circledcirc \mathbf{\Omega}) \circledcirc \mathbf{e}_1 = \mathbf{e}_{31} \circledcirc \mathbf{e}_1 = \alpha_{3\bar{1}} \mathbf{e}_3 \\
&\mathbf{e}_2 \circledcirc (\mathbf{e}_1 \circledcirc \mathbf{\Omega}) = \mathbf{e}_2 \circledcirc \mathbf{e}_{23} = \alpha_{\bar{2}3} \mathbf{e}_3 \\
&(\mathbf{e}_2 \circledcirc \mathbf{e}_1) \circledcirc \mathbf{\Omega} = -\mathbf{e}_{12} \circledcirc \mathbf{\Omega} = -\mathbf{e}_3.
\end{aligned} \tag{5.12}
$$

Using Axiom VIII.2, we can relate the very last expressions from the six lines in equation sets 5.11 and 5.12 above to obtain:

$$\alpha_{3\bar{2}} = -\alpha_{\bar{2}3} = 1,$$
$$\alpha_{\bar{1}3} = -\alpha_{3\bar{1}} = 1.$$

The remaining values for the α constants can be obtained from the results in above equation set by making the following reverse, $1 \to 3$, $2 \to 1$, $3 \to 2$, and forward, $1 \to 2$, $2 \to 3$, $3 \to 1$, cyclic substitutions in the indices of the constants. The result of this process combined with the result above is the following set of values for all the α constants:

$$\begin{aligned} \alpha_{3\bar{2}} &= -\alpha_{\bar{2}3} = 1, \\ \alpha_{\bar{1}3} &= -\alpha_{3\bar{1}} = 1, \end{aligned} \tag{5.13a}$$

$$\begin{aligned} \alpha_{2\bar{1}} &= -\alpha_{\bar{1}2} = 1, \\ \alpha_{\bar{3}2} &= -\alpha_{2\bar{3}} = 1, \end{aligned} \tag{5.13b}$$

$$\begin{aligned} \alpha_{1\bar{3}} &= -\alpha_{\bar{3}1} = 1, \\ \alpha_{\bar{2}1} &= -\alpha_{1\bar{2}} = 1. \end{aligned} \tag{5.13c}$$

Now we are ready to determine the values of the β constants. These constants are associated with products of bivector factors. The β's are multi-indexed by the surviving integers of the two bivector factors in a given product. The order of the integers in any β constant's multi-index is the same as that of the order of the bivector factors from which they were "inherited."

Consider now the following products which follow the pattern of Axiom VIII.2:

$$\begin{aligned} (\mathbf{e}_{31} \circledcirc \boldsymbol{\Omega}) \circledcirc \mathbf{e}_{12} &= \mathbf{e}_2 \circledcirc \mathbf{e}_{12} = \alpha_{\bar{2}1}\mathbf{e}_1 = \mathbf{e}_1 \\ \mathbf{e}_{31} \circledcirc (\mathbf{e}_{12} \circledcirc \boldsymbol{\Omega}) &= \mathbf{e}_{31} \circledcirc \mathbf{e}_3 = \alpha_{1\bar{3}}\mathbf{e}_1 = \mathbf{e}_1 \\ (\mathbf{e}_{31} \circledcirc \mathbf{e}_{12}) \circledcirc \boldsymbol{\Omega} &= \beta_{32}\mathbf{e}_{23} \circledcirc \boldsymbol{\Omega} = \beta_{32}\mathbf{e}_1. \end{aligned} \tag{5.14}$$

Commuting $\mathbf{e}_{31}$ and $\mathbf{e}_{12}$ in the above set of equations leads to the following set:

$$\begin{aligned} (\mathbf{e}_{12} \circledcirc \boldsymbol{\Omega}) \circledcirc \mathbf{e}_{31} &= \mathbf{e}_3 \circledcirc \mathbf{e}_{31} = \alpha_{\bar{3}1}\mathbf{e}_1 = -\mathbf{e}_1 \\ \mathbf{e}_{12} \circledcirc (\mathbf{e}_{31} \circledcirc \boldsymbol{\Omega}) &= \mathbf{e}_{12} \circledcirc \mathbf{e}_2 = \alpha_{1\bar{2}}\mathbf{e}_1 = -\mathbf{e}_1 \\ (\mathbf{e}_{12} \circledcirc \mathbf{e}_{31}) \circledcirc \boldsymbol{\Omega} &= \beta_{23}\mathbf{e}_{23} \circledcirc \boldsymbol{\Omega} = \beta_{23}\mathbf{e}_1. \end{aligned} \tag{5.15}$$

Using Axiom VIII.2, we can relate the very last expressions from the six lines in equation sets 5.14 and 5.15 above to obtain:

$$\beta_{32} = -\beta_{23} = 1. \tag{5.16}$$

The remaining values for the β constants can be obtained from the results in equation set 5.16 by making the following reverse, $1 \to 3$, $2 \to 1$, $3 \to 2$, and forward, $1 \to 2$, $2 \to 3$, $3 \to 1$, cyclic substitutions in the indices of the constants. The result of this process combined with the result above is the following set of values for all the β constants:

$$\begin{aligned} \beta_{32} &= -\beta_{23} = 1, \\ \beta_{21} &= -\beta_{12} = 1, \\ \beta_{13} &= -\beta_{31} = 1. \end{aligned} \tag{5.17}$$

The complete multiplication table for $\mathcal{OC}_3$ is derived from Tables 5.1 and 5.2 by substituting the value 1 for all multi-indexed γ constants, and by substituting the

Table 5.1. Derivation of the Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_3$: I. The entries in the red tinted cells have been derived using scalar multiplication and Axiom Set VII′.

$a \circledcirc b$	b: 1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{\Omega}$
a: 1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{\Omega}$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\alpha_{\bar{1}2}\mathbf{e}_2$	$\alpha_{\bar{1}3}\mathbf{e}_3$	$\mathbf{\Omega}$	$\gamma_1\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$\alpha_{\bar{2}1}\mathbf{e}_1$	$\mathbf{\Omega}$	$\alpha_{\bar{2}3}\mathbf{e}_3$	$\gamma_2\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\mathbf{\Omega}$	$\alpha_{\bar{3}1}\mathbf{e}_1$	$\alpha_{\bar{3}2}\mathbf{e}_2$	$\gamma_3\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\alpha_{2\bar{1}}\mathbf{e}_2$	$\alpha_{1\bar{2}}\mathbf{e}_1$	$\mathbf{\Omega}$	$\underline{1}$	$\beta_{23}\mathbf{e}_{23}$	$\beta_{13}\mathbf{e}_{31}$	$\gamma_{12}\mathbf{e}_3$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\alpha_{3\bar{1}}\mathbf{e}_3$	$\mathbf{\Omega}$	$\alpha_{1\bar{3}}\mathbf{e}_1$	$\beta_{32}\mathbf{e}_{23}$	$\underline{1}$	$\beta_{12}\mathbf{e}_{12}$	$\gamma_{31}\mathbf{e}_2$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\mathbf{\Omega}$	$\alpha_{3\bar{2}}\,\mathbf{e}_3$	$\alpha_{2\bar{3}}\,\mathbf{e}_2$	$\beta_{31}\mathbf{e}_{31}$	$\beta_{21}\mathbf{e}_{12}$	$\underline{1}$	$\gamma_{23}\mathbf{e}_1$
$\mathbf{\Omega}$	$\mathbf{\Omega}$	$\gamma_1\mathbf{e}_{23}$	$\gamma_2\mathbf{e}_{31}$	$\gamma_3\mathbf{e}_{12}$	$\gamma_{12}\mathbf{e}_3$	$\gamma_{31}\mathbf{e}_2$	$\gamma_{23}\mathbf{e}_1$	$\underline{1}$

values taken from Equation sets 5.13 and 5.17 above for all the multi-indexed α and β constants. It is presented as Table 5.6 of the next subsection.

5.8. **More Multiplication Tables.** We end with multiplication tables for the Clifford algebras $\mathcal{C}\ell_2$ (Table 5.3), $\mathcal{C}\ell_3$ (Tables 5.5, 5.7), and for the orientation congruent algebras $\mathcal{OC}_2$ (Table 5.4), $\mathcal{OC}_3$ (Tables 5.6, 5.8), $\mathcal{OC}_4$ (Table 5.9), and $\mathcal{OC}_5$ (Table 5.10). In these tables we write the pseudoscalars of the two Clifford algebras $\mathcal{C}\ell_2$ and $\mathcal{C}\ell_3$ as $\mathbf{I} = \mathbf{e}_{12} = \mathbf{e}_1 \circ \mathbf{e}_2$ and $\mathbf{I} = \mathbf{e}_{123} = \mathbf{e}_1 \circ \mathbf{e}_2 \circ \mathbf{e}_3$. Also, we write here the perfect counits of the two perfect orientation congruent algebras $\mathcal{OC}_3$ and $\mathcal{OC}_5$ in omega notation as $\mathbf{\Omega} = \mathbf{e}_{123} = \mathbf{e}_1 \circledcirc \mathbf{e}_2 \circledcirc \mathbf{e}_3$ and $\mathbf{\Omega} = \mathbf{e}_{12345} = \mathbf{e}_1 \circledcirc \mathbf{e}_2 \circledcirc \mathbf{e}_3 \circledcirc \mathbf{e}_4 \circledcirc \mathbf{e}_5$. Finally, we write here the imperfect counits of the two imperfect orientation congruent algebras $\mathcal{OC}_2$ and $\mathcal{OC}_4$ in inverted omega notation as $\boldsymbol{\mho} = \mathbf{e}_{12} = \mathbf{e}_1 \circledcirc \mathbf{e}_2$ and $\boldsymbol{\mho} = \mathbf{e}_{1234} = \mathbf{e}_1 \circledcirc \mathbf{e}_2 \circledcirc \mathbf{e}_3 \circledcirc \mathbf{e}_4$.

The underlined entries in the orientation congruent algebra multiplication tables are oppositely signed compared to those in the tables for the corresponding Clifford algebras. We adopt this convenient underlining here, even though it conflicts with the convention mentioned in Definition 5.20 of subsection 5.4 which uses underlining of the symbol $\mathbf{\Omega}$ to distinguish *the* counit. Also, in all tables the entries in red-tinted cells are negatively signed; while the entries in untinted cells are positively signed.

Tables 5.6 and 5.10 show a certain form of the multiplication tables for the algebras $\mathcal{OC}_3$ and $\mathcal{OC}_5$. The cell coloring in these tables makes the reflection symmetry of the signs of the products about the central horizontal and vertical axes easy to see. Table 5.5 shows the same form of the multiplication table for the Clifford algebra $\mathcal{C}\ell_3$. Here the pattern of cell coloring has no obvious symmetry.

TABLE 5.2. Derivation of the Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_3$: II. The entries in the red tinted cells have already been derived. The entries in the remaining untinted cells are derived using Axiom Set VIII.

	$a \circledcirc b$	b: 1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
a	1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
	$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\alpha_{\bar{1}2}\mathbf{e}_2$	$\alpha_{\bar{1}3}\mathbf{e}_3$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$\alpha_{\bar{2}1}\mathbf{e}_1$	$\boldsymbol{\Omega}$	$\alpha_{\bar{2}3}\mathbf{e}_3$	$\mathbf{e}_{31}$
	$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\boldsymbol{\Omega}$	$\alpha_{\bar{3}1}\mathbf{e}_1$	$\alpha_{\bar{3}2}\mathbf{e}_2$	$\mathbf{e}_{12}$
	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\alpha_{2\bar{1}}\mathbf{e}_2$	$\alpha_{1\bar{2}}\mathbf{e}_1$	$\boldsymbol{\Omega}$	$\underline{1}$	$\beta_{23}\mathbf{e}_{23}$	$\beta_{13}\mathbf{e}_{31}$	$\mathbf{e}_3$
	$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\alpha_{3\bar{1}}\mathbf{e}_3$	$\boldsymbol{\Omega}$	$\alpha_{1\bar{3}}\mathbf{e}_1$	$\beta_{32}\mathbf{e}_{23}$	$\underline{1}$	$\beta_{12}\mathbf{e}_{12}$	$\mathbf{e}_2$
	$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$	$\alpha_{3\bar{2}}\mathbf{e}_3$	$\alpha_{2\bar{3}}\mathbf{e}_2$	$\beta_{31}\mathbf{e}_{31}$	$\beta_{21}\mathbf{e}_{12}$	$\underline{1}$	$\mathbf{e}_1$
	$\boldsymbol{\Omega}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_1$	1

Both the reflection symmetries in Tables 5.6 and 5.10 and their lack in Table 5.5 result from displaying these tables in a canonical form specific to the orientation congruent algebra. The arrangement of these tables is an example of a *multiplication table canonical form* (MTCF) of type *OC1*.

Any MTCF for an algebra is determined by just two criteria:

(1) the ordering chosen for the multi-indices of each basis blade; and
(2) the ordering of the basis blades in the indicial leftmost column and top row of the table.

I have not yet worked out the general definition of a MTCF of type OC1 and am deferring full investigation of this combinatorial problem until a later publication. Nevertheless, we may still roughly say that a type OC1 MTCF satisfies the type 1 criterion above by ordering the multi-indices of the basis blades so that as a set they are *coherently oriented* (in a specific way) relative to the counit $\underline{\boldsymbol{\Omega}}$. Also, we may roughly say that it satisfies the type 2 criterion above by placing the factor basis blades in the indicial column and row in a kind of *graded, reflected complementary order.* As the "1" in "OC1" suggests these two requirements define just one of several related multiplication table canonical forms.

For Clifford algebras we can define a MTCF of type *CL1* that is specified by *increasing numerical order* within the multi-index sequences of each basis blade and *Gray code order* [123, pp. 281 ff.] for the factor basis blades in the indicial column and row. Tables 5.7 and 5.8 contain the multiplication tables of the Clifford algebra $\mathcal{C}\ell_3$ and the orientation congruent algebra $\mathcal{OC}_3$ in CL1 form. If a Clifford algebra multiplication table is in CL1 canonical form, the signs of the products display reflection symmetry about the central vertical axis just as they do for an orientation congruent algebra multiplication table in OC1 form. However, the second sign symmetry pattern differs: it becomes vertical *translation* symmetry between adjacent rows paired off starting from the first. And now it is the $\mathcal{OC}_3$ multiplication table in CL1 canonical form whose product signs display no obvious symmetries.

Table 5.3. The Multiplication Table for the Orientation Congruent Algebra $\mathcal{C}\ell_2$. Red cell entries are negative.

		b			
	$a \circ b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{I}$
a	1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{I}$
	$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{I}$	$\mathbf{e}_2$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{I}$	1	$-\mathbf{e}_1$
	$\mathbf{I}$	$\mathbf{I}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

Table 5.4. The Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_2$. Red cell entries are negative.

		b			
	$a \circledcirc b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\boldsymbol{\mho}$
a	1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\boldsymbol{\mho}$
	$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\boldsymbol{\mho}$	$-\mathbf{e}_2$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$-\boldsymbol{\mho}$	1	$\mathbf{e}_1$
	$\boldsymbol{\mho}$	$\boldsymbol{\mho}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	1

TABLE 5.5. The Multiplication Table for the Clifford Algebra $\mathcal{C}\ell_3$. The factors are in reflected, complementary grade order with indices in orientation congruent order. Red cell entries are negative.

	$a \circ b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{I}$
a	1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{I}$
	$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$\mathbf{I}$	$\mathbf{e}_{23}$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_1$	$\mathbf{I}$	$\mathbf{e}_3$	$\mathbf{e}_{31}$
	$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\mathbf{I}$	$\mathbf{e}_1$	$-\mathbf{e}_2$	$\mathbf{e}_{12}$
	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{I}$	-1	$\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_3$
	$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$\mathbf{I}$	$-\mathbf{e}_1$	$-\mathbf{e}_{23}$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_2$
	$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\mathbf{I}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	-1	$-\mathbf{e}_1$
	$\mathbf{I}$	$\mathbf{I}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	-1

(Column heading: b)

TABLE 5.6. The Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_3$. The factors and indices are ordered as in Table 5.5 above. Red cell entries are negative. Underlined entries are signed oppositely to those in Table 5.5.

	$a \circledcirc b$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
a	1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$
	$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\underline{-\mathbf{e}_2}$	$\underline{\mathbf{e}_3}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$\underline{\mathbf{e}_1}$	$\boldsymbol{\Omega}$	$\underline{-\mathbf{e}_3}$	$\mathbf{e}_{31}$
	$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\boldsymbol{\Omega}$	$\underline{-\mathbf{e}_1}$	$\underline{\mathbf{e}_2}$	$\mathbf{e}_{12}$
	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\underline{\mathbf{e}_2}$	$\underline{-\mathbf{e}_1}$	$\boldsymbol{\Omega}$	$\underline{1}$	$\underline{-\mathbf{e}_{23}}$	$\underline{\mathbf{e}_{31}}$	$\underline{\mathbf{e}_3}$
	$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\underline{-\mathbf{e}_3}$	$\boldsymbol{\Omega}$	$\underline{\mathbf{e}_1}$	$\underline{\mathbf{e}_{23}}$	$\underline{1}$	$\underline{-\mathbf{e}_{12}}$	$\underline{\mathbf{e}_2}$
	$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$	$\underline{\mathbf{e}_3}$	$\underline{-\mathbf{e}_2}$	$\underline{-\mathbf{e}_{31}}$	$\underline{\mathbf{e}_{12}}$	$\underline{1}$	$\underline{\mathbf{e}_1}$
	$\boldsymbol{\Omega}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\underline{\mathbf{e}_3}$	$\underline{\mathbf{e}_2}$	$\underline{\mathbf{e}_1}$	$\underline{1}$

(Column heading: b)

TABLE 5.7. The Multiplication Table for the Clifford Algebra $\mathcal{C}\ell_3$. The factors are in Gray code order with indices in increasing numerical order. Red cell entries are negative.

		b							
	$a \circ b$	1	$\mathbf{e}_1$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$\mathbf{e}_{23}$	$\mathbf{I}$	$\mathbf{e}_{13}$	$\mathbf{e}_3$
a	1	1	$\mathbf{e}_1$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$\mathbf{e}_{23}$	$\mathbf{I}$	$\mathbf{e}_{13}$	$\mathbf{e}_3$
	$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_2$	$\mathbf{e}_{12}$	$\mathbf{I}$	$\mathbf{e}_{23}$	$\mathbf{e}_3$	$\mathbf{e}_{13}$
	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	-1	$\mathbf{e}_1$	$\mathbf{e}_{13}$	$-\mathbf{e}_3$	$-\mathbf{e}_{23}$	$\mathbf{I}$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	$-\mathbf{e}_1$	1	$\mathbf{e}_3$	$-\mathbf{e}_{13}$	$-\mathbf{I}$	$\mathbf{e}_{23}$
	$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\mathbf{I}$	$-\mathbf{e}_{13}$	$-\mathbf{e}_3$	-1	$-\mathbf{e}_1$	$\mathbf{e}_{12}$	$\mathbf{e}_2$
	$\mathbf{I}$	$\mathbf{I}$	$\mathbf{e}_{23}$	$-\mathbf{e}_3$	$-\mathbf{e}_{13}$	$-\mathbf{e}_1$	-1	$\mathbf{e}_2$	$\mathbf{e}_{12}$
	$\mathbf{e}_{13}$	$\mathbf{e}_{13}$	$-\mathbf{e}_3$	$\mathbf{e}_{23}$	$-\mathbf{I}$	$-\mathbf{e}_{12}$	$\mathbf{e}_2$	-1	$\mathbf{e}_1$
	$\mathbf{e}_3$	$\mathbf{e}_3$	$-\mathbf{e}_{13}$	$\mathbf{I}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_2$	$\mathbf{e}_{12}$	$-\mathbf{e}_1$	1

TABLE 5.8. The Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_3$. The factors and indices are ordered as in Table 5.7 above. Red cell entries are negative. Underlined entries are signed oppositely to those in Table 5.7.

		b							
	$a \circledcirc b$	1	$\mathbf{e}_1$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{13}$	$\mathbf{e}_3$
a	1	1	$\mathbf{e}_1$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{13}$	$\mathbf{e}_3$
	$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\underline{-\mathbf{e}_2}$	$\mathbf{e}_{12}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$	$\underline{-\mathbf{e}_3}$	$\mathbf{e}_{13}$
	$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\underline{\mathbf{e}_2}$	$\underline{1}$	$\underline{-\mathbf{e}_1}$	$\underline{-\mathbf{e}_{13}}$	$\underline{\mathbf{e}_3}$	$\underline{\mathbf{e}_{23}}$	$\boldsymbol{\Omega}$
	$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	$\underline{\mathbf{e}_1}$	1	$\underline{-\mathbf{e}_3}$	$-\mathbf{e}_{13}$	$-\boldsymbol{\Omega}$	$\mathbf{e}_{23}$
	$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\boldsymbol{\Omega}$	$\underline{\mathbf{e}_{13}}$	$\underline{\mathbf{e}_3}$	$\underline{1}$	$\underline{\mathbf{e}_1}$	$\underline{-\mathbf{e}_{12}}$	$\underline{-\mathbf{e}_2}$
	$\boldsymbol{\Omega}$	$\boldsymbol{\Omega}$	$\mathbf{e}_{23}$	$\underline{\mathbf{e}_3}$	$-\mathbf{e}_{13}$	$\underline{\mathbf{e}_1}$	$\underline{1}$	$\underline{-\mathbf{e}_2}$	$\mathbf{e}_{12}$
	$\mathbf{e}_{13}$	$\mathbf{e}_{13}$	$\underline{\mathbf{e}_3}$	$\underline{-\mathbf{e}_{23}}$	$-\boldsymbol{\Omega}$	$\underline{\mathbf{e}_{12}}$	$\underline{-\mathbf{e}_2}$	$\underline{1}$	$\underline{-\mathbf{e}_1}$
	$\mathbf{e}_3$	$\mathbf{e}_3$	$-\mathbf{e}_{13}$	$\boldsymbol{\Omega}$	$-\mathbf{e}_{23}$	$\underline{\mathbf{e}_2}$	$\mathbf{e}_{12}$	$\underline{\mathbf{e}_1}$	1

TABLE 5.9. The Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_4$. The indices of each factor are ordered as in Table 5.10 for the orientation congruent algebra $\mathcal{OC}_5$. Red cell entries are negative.

		b															
	$a \odot b$	1	e_{1}	e_{2}	e_{3}	e_{4}	e_{12}	e_{31}	e_{14}	e_{23}	e_{42}	e_{34}	e_{123}	e_{124}	e_{134}	e_{234}	$\mho$
a	1	1	e_{1}	e_{2}	e_{3}	e_{4}	e_{12}	e_{31}	e_{14}	e_{23}	e_{42}	e_{34}	e_{123}	e_{124}	e_{134}	e_{234}	$\mho$
	e_{1}	e_{1}	1	e_{12}	$-e_{31}$	e_{14}	$-e_{2}$	e_{3}	$-e_{4}$	e_{123}	$-e_{124}$	e_{134}	e_{23}	$-e_{42}$	e_{34}	$\mho$	$-e_{234}$
	e_{2}	e_{2}	$-e_{12}$	1	e_{23}	$-e_{42}$	e_{1}	e_{123}	$-e_{124}$	$-e_{3}$	e_{4}	e_{234}	e_{31}	$-e_{14}$	$-\mho$	e_{34}	e_{134}
	e_{3}	e_{3}	e_{31}	$-e_{23}$	1	e_{34}	e_{123}	$-e_{1}$	$-e_{134}$	e_{2}	e_{234}	$-e_{4}$	e_{12}	$\mho$	$-e_{14}$	e_{42}	$-e_{124}$
	e_{4}	e_{4}	$-e_{14}$	e_{42}	$-e_{34}$	1	e_{124}	$-e_{134}$	e_{1}	e_{234}	$-e_{2}$	e_{3}	$-\mho$	e_{12}	$-e_{31}$	e_{23}	e_{123}
	e_{12}	e_{12}	e_{2}	$-e_{1}$	e_{123}	e_{124}	1	$-e_{23}$	$-e_{42}$	e_{31}	e_{14}	$\mho$	e_{3}	e_{4}	e_{234}	$-e_{134}$	e_{34}
	e_{31}	e_{31}	$-e_{3}$	e_{123}	e_{1}	$-e_{134}$	e_{23}	1	$-e_{34}$	$-e_{12}$	$-\mho$	e_{14}	e_{2}	e_{234}	$-e_{4}$	$-e_{124}$	$-e_{42}$
	e_{14}	e_{14}	e_{4}	$-e_{124}$	$-e_{134}$	$-e_{1}$	e_{42}	e_{34}	1	$\mho$	$-e_{12}$	$-e_{31}$	e_{234}	$-e_{2}$	$-e_{3}$	$-e_{123}$	e_{23}
	e_{23}	e_{23}	e_{123}	e_{3}	$-e_{2}$	e_{234}	$-e_{31}$	e_{12}	$\mho$	1	$-e_{34}$	e_{42}	e_{1}	e_{134}	$-e_{124}$	e_{4}	e_{14}
	e_{42}	e_{42}	$-e_{124}$	$-e_{4}$	e_{234}	e_{2}	$-e_{14}$	$-\mho$	e_{12}	e_{34}	1	$-e_{23}$	e_{134}	$-e_{1}$	$-e_{123}$	e_{3}	$-e_{31}$
	e_{34}	e_{34}	e_{134}	e_{234}	e_{4}	$-e_{3}$	$\mho$	$-e_{14}$	e_{31}	$-e_{42}$	e_{23}	1	e_{124}	$-e_{123}$	e_{1}	e_{2}	e_{12}
	e_{123}	e_{123}	e_{23}	e_{31}	e_{12}	$\mho$	e_{3}	e_{2}	$-e_{234}$	e_{1}	$-e_{134}$	$-e_{124}$	1	e_{34}	e_{42}	e_{14}	$-e_{4}$
	e_{124}	e_{124}	$-e_{42}$	$-e_{14}$	$-\mho$	e_{12}	e_{4}	$-e_{234}$	$-e_{2}$	$-e_{134}$	$-e_{1}$	e_{123}	$-e_{34}$	1	e_{23}	e_{31}	e_{3}
	e_{134}	e_{134}	e_{34}	$\mho$	$-e_{14}$	$-e_{31}$	$-e_{234}$	$-e_{4}$	$-e_{3}$	e_{124}	e_{123}	e_{1}	$-e_{42}$	$-e_{23}$	1	e_{12}	$-e_{2}$
	e_{234}	e_{234}	$-\mho$	e_{34}	e_{42}	e_{23}	e_{134}	e_{124}	e_{123}	e_{4}	e_{3}	e_{2}	$-e_{14}$	$-e_{31}$	$-e_{12}$	1	e_{1}
	$\mho$	$\mho$	e_{234}	$-e_{134}$	e_{124}	$-e_{123}$	e_{34}	$-e_{42}$	e_{23}	e_{14}	$-e_{31}$	e_{12}	e_{4}	$-e_{3}$	e_{2}	$-e_{1}$	1

Table 5.10: The Multiplication Table for the Orientation Congruent Algebra $\mathcal{OC}_5$

b (columns), a (rows)

$a \circledcirc b$	1	e_{1}	e_{2}	e_{3}	e_{4}	e_{5}	e_{34}	e_{25}	e_{45}	e_{31}	e_{51}	e_{42}	e_{12}	e_{53}	e_{23}	e_{14}	e_{235}	e_{145}	e_{124}	e_{345}	e_{135}	e_{234}	e_{245}	e_{123}	e_{134}	e_{125}	e_{1234}	e_{1253}	e_{1245}	e_{3145}	e_{2345}	Ω
1	1	e_{1}	e_{2}	e_{3}	e_{4}	e_{5}	e_{34}	e_{25}	e_{45}	e_{31}	e_{51}	e_{42}	e_{12}	e_{53}	e_{23}	e_{14}	e_{235}	e_{145}	e_{124}	e_{345}	e_{135}	e_{234}	e_{245}	e_{123}	e_{134}	e_{125}	e_{1234}	e_{1253}	e_{1245}	e_{3145}	e_{2345}	Ω
e_{1}	e_{1}	1	e_{12}	$-e_{31}$	e_{14}	$-e_{51}$	e_{134}	e_{125}	e_{145}	$\underline{e_{3}}$	$\underline{e_{5}}$	$-e_{124}$	$-\underline{e_{2}}$	$-e_{135}$	e_{123}	$-\underline{e_{4}}$	$-e_{1253}$	e_{45}	$-e_{42}$	$-e_{3145}$	$-e_{53}$	e_{1234}	e_{1245}	e_{23}	e_{34}	e_{25}	$-\underline{e_{234}}$	$\underline{e_{235}}$	$-\underline{e_{245}}$	$\underline{e_{345}}$	Ω	e_{2345}
e_{2}	e_{2}	$-e_{12}$	1	e_{23}	$-e_{42}$	e_{25}	e_{234}	$-\underline{e_{5}}$	e_{245}	e_{123}	e_{125}	$\underline{e_{4}}$	$\underline{e_{1}}$	$-e_{235}$	$-\underline{e_{3}}$	$-e_{124}$	$-e_{53}$	$-e_{1245}$	$-e_{14}$	e_{2345}	e_{1253}	e_{34}	e_{45}	e_{31}	$-e_{1234}$	e_{51}	$\underline{e_{134}}$	$-\underline{e_{135}}$	$\underline{e_{145}}$	Ω	$-\underline{e_{345}}$	e_{3145}
e_{3}	e_{3}	e_{31}	$-e_{23}$	1	e_{34}	$-e_{53}$	$-\underline{e_{4}}$	$-e_{235}$	e_{345}	$-\underline{e_{1}}$	e_{135}	e_{234}	e_{123}	$\underline{e_{5}}$	$\underline{e_{2}}$	$-e_{134}$	$-e_{25}$	e_{3145}	e_{1234}	e_{45}	e_{51}	e_{42}	$-e_{2345}$	e_{12}	$-e_{14}$	$-e_{1253}$	$-\underline{e_{124}}$	$\underline{e_{125}}$	Ω	$-\underline{e_{145}}$	$\underline{e_{245}}$	e_{1245}
e_{4}	e_{4}	$-e_{14}$	e_{42}	$-e_{34}$	1	e_{45}	$\underline{e_{3}}$	$-e_{245}$	$-\underline{e_{5}}$	$-e_{134}$	e_{145}	$-\underline{e_{2}}$	e_{124}	e_{345}	e_{234}	$\underline{e_{1}}$	e_{2345}	e_{51}	e_{12}	e_{53}	$-e_{3145}$	e_{23}	$-e_{25}$	$-e_{1234}$	$-e_{31}$	e_{1245}	$\underline{e_{123}}$	Ω	$-\underline{e_{125}}$	$\underline{e_{135}}$	$-\underline{e_{235}}$	e_{1253}
e_{5}	e_{5}	e_{51}	$-e_{25}$	e_{53}	$-e_{45}$	1	e_{345}	$\underline{e_{2}}$	$\underline{e_{4}}$	$-e_{135}$	$-\underline{e_{1}}$	$-e_{245}$	e_{125}	$-\underline{e_{3}}$	e_{235}	e_{145}	e_{23}	e_{14}	$-e_{1245}$	e_{34}	$-e_{31}$	$-e_{2345}$	$-e_{42}$	e_{1253}	e_{3145}	e_{12}	Ω	$-\underline{e_{123}}$	$\underline{e_{124}}$	$-\underline{e_{134}}$	$\underline{e_{234}}$	e_{1234}
e_{34}	e_{34}	e_{134}	e_{234}	$\underline{e_{4}}$	$-\underline{e_{3}}$	e_{345}	$\underline{1}$	e_{2345}	$\underline{e_{53}}$	$-\underline{e_{14}}$	e_{3145}	$\underline{e_{23}}$	e_{1234}	$-\underline{e_{45}}$	$-\underline{e_{42}}$	$\underline{e_{31}}$	$\underline{e_{245}}$	$-\underline{e_{135}}$	$-\underline{e_{123}}$	$\underline{e_{5}}$	$\underline{e_{145}}$	$\underline{e_{2}}$	$-\underline{e_{235}}$	$\underline{e_{124}}$	$\underline{e_{1}}$	Ω	$\underline{e_{12}}$	$-\underline{e_{1245}}$	$\underline{e_{1253}}$	$\underline{e_{51}}$	$\underline{e_{25}}$	$\underline{e_{125}}$
e_{25}	e_{25}	e_{125}	$\underline{e_{5}}$	$-e_{235}$	$-e_{245}$	$-\underline{e_{2}}$	e_{2345}	$\underline{1}$	$-\underline{e_{42}}$	$-e_{1253}$	$\underline{e_{12}}$	$\underline{e_{45}}$	$-\underline{e_{51}}$	$-\underline{e_{23}}$	$\underline{e_{53}}$	$-e_{1245}$	$-\underline{e_{3}}$	$\underline{e_{124}}$	$-\underline{e_{145}}$	$-\underline{e_{234}}$	$\underline{e_{123}}$	$\underline{e_{345}}$	$-\underline{e_{4}}$	$-\underline{e_{135}}$	Ω	$\underline{e_{1}}$	$-\underline{e_{3145}}$	$-\underline{e_{31}}$	$-\underline{e_{14}}$	$\underline{e_{1234}}$	$\underline{e_{34}}$	$\underline{e_{134}}$
e_{45}	e_{45}	e_{145}	e_{245}	e_{345}	$\underline{e_{5}}$	$-\underline{e_{4}}$	$-\underline{e_{53}}$	$\underline{e_{42}}$	$\underline{1}$	e_{3145}	$\underline{e_{14}}$	$-\underline{e_{25}}$	e_{1245}	$\underline{e_{34}}$	e_{2345}	$-\underline{e_{51}}$	$-\underline{e_{234}}$	$\underline{e_{1}}$	$\underline{e_{125}}$	$\underline{e_{3}}$	$-\underline{e_{134}}$	$\underline{e_{235}}$	$\underline{e_{2}}$	Ω	$\underline{e_{135}}$	$-\underline{e_{124}}$	$-\underline{e_{1253}}$	$\underline{e_{1234}}$	$\underline{e_{12}}$	$\underline{e_{31}}$	$\underline{e_{23}}$	$\underline{e_{123}}$
e_{31}	e_{31}	$-\underline{e_{3}}$	e_{123}	$\underline{e_{1}}$	$-e_{134}$	$-e_{135}$	$\underline{e_{14}}$	$-e_{1253}$	e_{3145}	$\underline{1}$	$-\underline{e_{53}}$	$-e_{1234}$	$\underline{e_{23}}$	$\underline{e_{51}}$	$-\underline{e_{12}}$	$-\underline{e_{34}}$	$-\underline{e_{125}}$	$-\underline{e_{345}}$	$\underline{e_{234}}$	$\underline{e_{145}}$	$-\underline{e_{5}}$	$-\underline{e_{124}}$	Ω	$\underline{e_{2}}$	$-\underline{e_{4}}$	$\underline{e_{235}}$	$-\underline{e_{42}}$	$-\underline{e_{25}}$	$\underline{e_{2345}}$	$\underline{e_{45}}$	$-\underline{e_{1245}}$	$\underline{e_{245}}$
e_{51}	e_{51}	$-\underline{e_{5}}$	e_{125}	e_{135}	e_{145}	$\underline{e_{1}}$	e_{3145}	$-\underline{e_{12}}$	$-\underline{e_{14}}$	$\underline{e_{53}}$	$\underline{1}$	e_{1245}	$\underline{e_{25}}$	$-\underline{e_{31}}$	e_{1253}	$\underline{e_{45}}$	$\underline{e_{123}}$	$\underline{e_{4}}$	$-\underline{e_{245}}$	$\underline{e_{134}}$	$\underline{e_{3}}$	Ω	$\underline{e_{124}}$	$-\underline{e_{235}}$	$-\underline{e_{345}}$	$\underline{e_{2}}$	$\underline{e_{2345}}$	$\underline{e_{23}}$	$\underline{e_{42}}$	$\underline{e_{34}}$	$-\underline{e_{1234}}$	$\underline{e_{234}}$
e_{42}	e_{42}	$-e_{124}$	$-\underline{e_{4}}$	e_{234}	$\underline{e_{2}}$	$-e_{245}$	$-\underline{e_{23}}$	$-\underline{e_{45}}$	$\underline{e_{25}}$	$-e_{1234}$	e_{1245}	$\underline{1}$	$-\underline{e_{14}}$	$-e_{2345}$	$\underline{e_{34}}$	$\underline{e_{12}}$	$\underline{e_{345}}$	$\underline{e_{125}}$	$-\underline{e_{1}}$	$-\underline{e_{235}}$	Ω	$\underline{e_{3}}$	$-\underline{e_{5}}$	$\underline{e_{134}}$	$-\underline{e_{123}}$	$-\underline{e_{145}}$	$-\underline{e_{31}}$	$\underline{e_{3145}}$	$\underline{e_{51}}$	$-\underline{e_{1253}}$	$-\underline{e_{53}}$	$\underline{e_{135}}$
e_{12}	e_{12}	$\underline{e_{2}}$	$-\underline{e_{1}}$	e_{123}	e_{124}	e_{125}	e_{1234}	$\underline{e_{51}}$	e_{1245}	$-\underline{e_{23}}$	$-\underline{e_{25}}$	$\underline{e_{14}}$	$\underline{1}$	e_{1253}	$\underline{e_{31}}$	$-\underline{e_{42}}$	$-\underline{e_{135}}$	$\underline{e_{245}}$	$\underline{e_{4}}$	Ω	$\underline{e_{235}}$	$-\underline{e_{134}}$	$-\underline{e_{145}}$	$\underline{e_{3}}$	$\underline{e_{234}}$	$\underline{e_{5}}$	$\underline{e_{34}}$	$\underline{e_{53}}$	$\underline{e_{45}}$	$-\underline{e_{2345}}$	$\underline{e_{3145}}$	$\underline{e_{345}}$
e_{53}	e_{53}	$-e_{135}$	$-e_{235}$	$-\underline{e_{5}}$	e_{345}	$\underline{e_{3}}$	$\underline{e_{45}}$	$\underline{e_{23}}$	$-\underline{e_{34}}$	$-\underline{e_{51}}$	$\underline{e_{31}}$	$-e_{2345}$	e_{1253}	$\underline{1}$	$-\underline{e_{25}}$	$-e_{3145}$	$-\underline{e_{2}}$	$-\underline{e_{134}}$	Ω	$\underline{e_{4}}$	$-\underline{e_{1}}$	$\underline{e_{245}}$	$-\underline{e_{234}}$	$-\underline{e_{125}}$	$\underline{e_{145}}$	$\underline{e_{123}}$	$\underline{e_{1245}}$	$\underline{e_{12}}$	$-\underline{e_{1234}}$	$-\underline{e_{14}}$	$-\underline{e_{42}}$	$\underline{e_{124}}$
e_{23}	e_{23}	e_{123}	$\underline{e_{3}}$	$-\underline{e_{2}}$	e_{234}	e_{235}	$\underline{e_{42}}$	$-\underline{e_{53}}$	e_{2345}	$\underline{e_{12}}$	e_{1253}	$-\underline{e_{34}}$	$-\underline{e_{31}}$	$\underline{e_{25}}$	$\underline{1}$	e_{1234}	$\underline{e_{5}}$	Ω	$\underline{e_{134}}$	$-\underline{e_{245}}$	$-\underline{e_{125}}$	$\underline{e_{4}}$	$\underline{e_{345}}$	$\underline{e_{1}}$	$-\underline{e_{124}}$	$\underline{e_{135}}$	$\underline{e_{14}}$	$\underline{e_{51}}$	$-\underline{e_{3145}}$	$\underline{e_{1245}}$	$\underline{e_{45}}$	$\underline{e_{145}}$
e_{14}	e_{14}	$\underline{e_{4}}$	$-e_{124}$	$-e_{134}$	$-\underline{e_{1}}$	e_{145}	$-\underline{e_{31}}$	$-e_{1245}$	$\underline{e_{51}}$	$\underline{e_{34}}$	$-\underline{e_{45}}$	$-\underline{e_{12}}$	$\underline{e_{42}}$	$-e_{3145}$	e_{1234}	$\underline{1}$	Ω	$\underline{e_{5}}$	$-\underline{e_{2}}$	$\underline{e_{135}}$	$-\underline{e_{345}}$	$-\underline{e_{123}}$	$\underline{e_{125}}$	$\underline{e_{234}}$	$-\underline{e_{3}}$	$-\underline{e_{245}}$	$\underline{e_{23}}$	$-\underline{e_{2345}}$	$-\underline{e_{25}}$	$-\underline{e_{53}}$	$\underline{e_{1253}}$	$\underline{e_{235}}$
e_{235}	e_{235}	e_{1253}	$-e_{53}$	$-e_{25}$	$-e_{2345}$	e_{23}	$-\underline{e_{245}}$	$-\underline{e_{3}}$	$\underline{e_{234}}$	$\underline{e_{125}}$	$-\underline{e_{123}}$	$-\underline{e_{345}}$	$\underline{e_{135}}$	$-\underline{e_{2}}$	$\underline{e_{5}}$	Ω	$\underline{1}$	e_{1234}	$-e_{3145}$	$\underline{e_{42}}$	$-\underline{e_{12}}$	$-\underline{e_{45}}$	$\underline{e_{34}}$	$\underline{e_{51}}$	$-e_{1245}$	$-\underline{e_{31}}$	$\underline{e_{145}}$	$-e_{1}$	$-\underline{e_{134}}$	$-\underline{e_{124}}$	e_{4}	$\underline{e_{14}}$
e_{145}	e_{145}	e_{45}	e_{1245}	$-e_{3145}$	e_{51}	e_{14}	$\underline{e_{135}}$	$-\underline{e_{124}}$	$\underline{e_{1}}$	$\underline{e_{345}}$	$\underline{e_{4}}$	$-\underline{e_{125}}$	$-\underline{e_{245}}$	$\underline{e_{134}}$	Ω	$\underline{e_{5}}$	e_{1234}	$\underline{1}$	$\underline{e_{25}}$	$-\underline{e_{31}}$	$-\underline{e_{34}}$	e_{1253}	$\underline{e_{12}}$	e_{2345}	$-\underline{e_{53}}$	$\underline{e_{42}}$	$\underline{e_{235}}$	$\underline{e_{234}}$	$-\underline{e_{2}}$	e_{3}	$\underline{e_{123}}$	$\underline{e_{23}}$
e_{124}	e_{124}	$-e_{42}$	$-e_{14}$	$-e_{1234}$	e_{12}	e_{1245}	$\underline{e_{123}}$	$\underline{e_{145}}$	$-\underline{e_{125}}$	$-\underline{e_{234}}$	$\underline{e_{245}}$	$-\underline{e_{1}}$	$\underline{e_{4}}$	Ω	$-\underline{e_{134}}$	$-\underline{e_{2}}$	$-e_{3145}$	$-\underline{e_{25}}$	$\underline{1}$	e_{1253}	$-\underline{e_{2345}}$	$\underline{e_{31}}$	$-\underline{e_{51}}$	$-\underline{e_{34}}$	$\underline{e_{23}}$	$\underline{e_{45}}$	e_{3}	$\underline{e_{345}}$	$-e_{5}$	$-\underline{e_{235}}$	$-\underline{e_{135}}$	$\underline{e_{53}}$
e_{345}	e_{345}	e_{3145}	$-e_{2345}$	e_{45}	e_{53}	e_{34}	$\underline{e_{5}}$	$\underline{e_{234}}$	$\underline{e_{3}}$	$-\underline{e_{145}}$	$-\underline{e_{134}}$	$\underline{e_{235}}$	Ω	$\underline{e_{4}}$	$\underline{e_{245}}$	$-\underline{e_{135}}$	$-\underline{e_{42}}$	$\underline{e_{31}}$	e_{1253}	$\underline{1}$	$\underline{e_{14}}$	$-\underline{e_{25}}$	$-\underline{e_{23}}$	e_{1245}	$\underline{e_{51}}$	e_{1234}	$\underline{e_{125}}$	$\underline{e_{124}}$	$\underline{e_{123}}$	$-e_{1}$	e_{2}	$\underline{e_{12}}$
e_{135}	e_{135}	$-e_{53}$	$-e_{1253}$	e_{51}	e_{3145}	$-e_{31}$	$-\underline{e_{145}}$	$-\underline{e_{123}}$	$\underline{e_{134}}$	$-\underline{e_{5}}$	$\underline{e_{3}}$	Ω	$-\underline{e_{235}}$	$-\underline{e_{1}}$	e_{125}	$\underline{e_{345}}$	$\underline{e_{12}}$	$\underline{e_{34}}$	$-e_{2345}$	$-\underline{e_{14}}$	$\underline{1}$	e_{1245}	$-e_{1234}$	$\underline{e_{25}}$	$-\underline{e_{45}}$	$-\underline{e_{23}}$	$-\underline{e_{245}}$	e_{2}	$\underline{e_{234}}$	$-\underline{e_{4}}$	$-\underline{e_{124}}$	$\underline{e_{42}}$
e_{234}	e_{234}	$-e_{1234}$	e_{34}	e_{42}	e_{23}	e_{2345}	$\underline{e_{2}}$	$-\underline{e_{345}}$	$-\underline{e_{235}}$	$\underline{e_{124}}$	Ω	$\underline{e_{3}}$	$\underline{e_{134}}$	$-\underline{e_{245}}$	$\underline{e_{4}}$	$\underline{e_{123}}$	$\underline{e_{45}}$	e_{1253}	$-\underline{e_{31}}$	$\underline{e_{25}}$	e_{1245}	$\underline{1}$	$\underline{e_{53}}$	$-\underline{e_{14}}$	$-\underline{e_{12}}$	e_{3145}	e_{1}	$\underline{e_{145}}$	$\underline{e_{135}}$	$\underline{e_{125}}$	$-e_{5}$	$\underline{e_{51}}$
e_{245}	e_{245}	$-e_{1245}$	e_{45}	e_{2345}	$-e_{25}$	$-e_{42}$	$\underline{e_{235}}$	$-\underline{e_{4}}$	$\underline{e_{2}}$	Ω	$-\underline{e_{124}}$	$-\underline{e_{5}}$	$\underline{e_{145}}$	$\underline{e_{234}}$	$-\underline{e_{345}}$	$-\underline{e_{125}}$	$-\underline{e_{34}}$	$-\underline{e_{12}}$	$\underline{e_{51}}$	$\underline{e_{23}}$	$-e_{1234}$	$-\underline{e_{53}}$	$\underline{1}$	e_{3145}	$-e_{1253}$	$\underline{e_{14}}$	$-\underline{e_{135}}$	$-\underline{e_{134}}$	e_{1}	$\underline{e_{123}}$	$-e_{3}$	$\underline{e_{31}}$
e_{123}	e_{123}	e_{23}	e_{31}	e_{12}	e_{1234}	$-e_{1253}$	$-\underline{e_{124}}$	$\underline{e_{135}}$	Ω	$\underline{e_{2}}$	$\underline{e_{235}}$	$-\underline{e_{134}}$	$\underline{e_{3}}$	$\underline{e_{125}}$	$\underline{e_{1}}$	$-\underline{e_{234}}$	$-\underline{e_{51}}$	e_{2345}	$\underline{e_{34}}$	e_{1245}	$-\underline{e_{25}}$	$\underline{e_{14}}$	e_{3145}	$\underline{1}$	$\underline{e_{42}}$	$-\underline{e_{53}}$	$-e_{4}$	e_{5}	$\underline{e_{345}}$	$\underline{e_{245}}$	$\underline{e_{145}}$	$\underline{e_{45}}$
e_{134}	e_{134}	e_{34}	e_{1234}	$-e_{14}$	$-e_{31}$	$-e_{3145}$	$\underline{e_{1}}$	Ω	$-\underline{e_{135}}$	$-\underline{e_{4}}$	$\underline{e_{345}}$	$\underline{e_{123}}$	$-\underline{e_{234}}$	$-\underline{e_{145}}$	$\underline{e_{124}}$	$-\underline{e_{3}}$	$-e_{1245}$	$\underline{e_{53}}$	$-\underline{e_{23}}$	$-\underline{e_{51}}$	$\underline{e_{45}}$	$\underline{e_{12}}$	$-e_{1253}$	$-\underline{e_{42}}$	$\underline{1}$	e_{2345}	$-e_{2}$	$-\underline{e_{245}}$	$-\underline{e_{235}}$	e_{5}	$\underline{e_{125}}$	$\underline{e_{25}}$
e_{125}	e_{125}	e_{25}	e_{51}	e_{1253}	$-e_{1245}$	e_{12}	Ω	$\underline{e_{1}}$	$\underline{e_{124}}$	$-\underline{e_{235}}$	$\underline{e_{2}}$	$\underline{e_{145}}$	$\underline{e_{5}}$	$-\underline{e_{123}}$	$-\underline{e_{135}}$	$\underline{e_{245}}$	$\underline{e_{31}}$	$-\underline{e_{42}}$	$-\underline{e_{45}}$	e_{1234}	$\underline{e_{23}}$	e_{3145}	$-\underline{e_{14}}$	$\underline{e_{53}}$	e_{2345}	$\underline{1}$	$\underline{e_{345}}$	$-e_{3}$	e_{4}	$\underline{e_{234}}$	$\underline{e_{134}}$	$\underline{e_{34}}$
e_{1234}	e_{1234}	$\underline{e_{234}}$	$-\underline{e_{134}}$	$\underline{e_{124}}$	$-\underline{e_{123}}$	Ω	$\underline{e_{12}}$	$\underline{e_{3145}}$	$\underline{e_{1253}}$	$-\underline{e_{42}}$	$-\underline{e_{2345}}$	$-\underline{e_{31}}$	$\underline{e_{34}}$	$-\underline{e_{1245}}$	$\underline{e_{14}}$	$\underline{e_{23}}$	$\underline{e_{145}}$	$\underline{e_{235}}$	$-e_{3}$	$\underline{e_{125}}$	$-\underline{e_{245}}$	$-e_{1}$	$-\underline{e_{135}}$	e_{4}	e_{2}	$\underline{e_{345}}$	1	$-e_{45}$	e_{53}	$-e_{25}$	e_{51}	e_{5}
e_{1253}	e_{1253}	$-\underline{e_{235}}$	$\underline{e_{135}}$	$-\underline{e_{125}}$	Ω	$\underline{e_{123}}$	e_{1245}	$-\underline{e_{31}}$	$-\underline{e_{1234}}$	$-\underline{e_{25}}$	$\underline{e_{23}}$	$-\underline{e_{3145}}$	$\underline{e_{53}}$	$\underline{e_{12}}$	$\underline{e_{51}}$	$\underline{e_{2345}}$	e_{1}	$\underline{e_{234}}$	$\underline{e_{345}}$	$\underline{e_{124}}$	$-e_{2}$	$\underline{e_{145}}$	$-\underline{e_{134}}$	$-e_{5}$	$-\underline{e_{245}}$	e_{3}	e_{45}	1	$-e_{34}$	e_{42}	$-e_{14}$	e_{4}
e_{1245}	e_{1245}	$\underline{e_{245}}$	$-\underline{e_{145}}$	Ω	$\underline{e_{125}}$	$-\underline{e_{124}}$	$-\underline{e_{1253}}$	$-\underline{e_{14}}$	$\underline{e_{12}}$	$-\underline{e_{2345}}$	$\underline{e_{42}}$	$\underline{e_{51}}$	$\underline{e_{45}}$	$\underline{e_{1234}}$	e_{3145}	$-\underline{e_{25}}$	$-\underline{e_{134}}$	e_{2}	e_{5}	$\underline{e_{123}}$	$\underline{e_{234}}$	$\underline{e_{135}}$	$-e_{1}$	$\underline{e_{345}}$	$-\underline{e_{235}}$	$-e_{4}$	$-e_{53}$	e_{34}	1	$-e_{23}$	e_{31}	e_{3}
e_{3145}	e_{3145}	$-\underline{e_{345}}$	Ω	$\underline{e_{145}}$	$-\underline{e_{135}}$	$\underline{e_{134}}$	$\underline{e_{51}}$	$-\underline{e_{1234}}$	$\underline{e_{31}}$	$\underline{e_{45}}$	$\underline{e_{34}}$	$\underline{e_{1253}}$	e_{2345}	$-\underline{e_{14}}$	$-\underline{e_{1245}}$	$-\underline{e_{53}}$	$-\underline{e_{124}}$	$-e_{3}$	$-\underline{e_{235}}$	e_{1}	e_{4}	$\underline{e_{125}}$	$\underline{e_{123}}$	$\underline{e_{245}}$	$-e_{5}$	$\underline{e_{234}}$	e_{25}	$-e_{42}$	e_{23}	1	$-e_{12}$	e_{2}
e_{2345}	e_{2345}	Ω	$\underline{e_{345}}$	$-\underline{e_{245}}$	$\underline{e_{235}}$	$-\underline{e_{234}}$	$\underline{e_{25}}$	$\underline{e_{34}}$	$\underline{e_{23}}$	$\underline{e_{1245}}$	$\underline{e_{1234}}$	$-\underline{e_{53}}$	$-\underline{e_{3145}}$	$-\underline{e_{42}}$	$\underline{e_{45}}$	$-\underline{e_{1253}}$	$-e_{4}$	$\underline{e_{123}}$	$-\underline{e_{135}}$	$-e_{2}$	$-\underline{e_{124}}$	e_{5}	e_{3}	$\underline{e_{145}}$	$\underline{e_{125}}$	$\underline{e_{134}}$	$-e_{51}$	e_{14}	$-e_{31}$	e_{12}	1	e_{1}
Ω	Ω	e_{2345}	e_{3145}	e_{1245}	e_{1253}	e_{1234}	$\underline{e_{125}}$	$\underline{e_{134}}$	$\underline{e_{123}}$	$\underline{e_{245}}$	$\underline{e_{234}}$	$\underline{e_{135}}$	$\underline{e_{345}}$	$\underline{e_{124}}$	$\underline{e_{145}}$	$\underline{e_{235}}$	$\underline{e_{14}}$	$\underline{e_{23}}$	$\underline{e_{53}}$	$\underline{e_{12}}$	$\underline{e_{42}}$	$\underline{e_{51}}$	$\underline{e_{31}}$	$\underline{e_{45}}$	$\underline{e_{25}}$	$\underline{e_{34}}$	e_{5}	e_{4}	e_{3}	e_{2}	e_{1}	1

6. The Clifford-Likeness of the Orientation Congruent Algebra

> Proof is an idol before whom the pure mathematician tortures himself.
>
> Sir Arthur Eddington [65, p. 337]

In this section we first give a fundamental defining formula for the sign factor function σ so that multiplying the Clifford product of basis blades by σ converts it into the orientation congruent product. Next, from that defining formula for the $\mathcal{OC}$ product of basis blades, we derive an expression for the $\mathcal{OC}$ product of general multivectors.

CHECK: Thus, we show that the orientation congruent algebra is a Clifford-like algebra.

Throughout the rest of the paper, we exploit the formula for the orientation congruent product based on the sign factor function in two ways. First, we use it theoretically to construct proofs. Second, we use it practically to calculate the orientation congruent product by hand or by computer.

In the last part of this section we validate of the sign factor function formula by proving

(1) that the orientation congruent product based on it satisfies the primed axioms given in the last section, and
(2) that the axioms for the orientation congruent algebra determine the sign factor function formula.

Actually, we explicitly prove (1) for only the last four axioms in Axiom Sets VII′ and VIII: Axioms VII.1′, VII.2′, VIII.1, and VIII.2. A proof for these four alone is sufficient because they are the only axioms that are either material modifications of some unprimed axiom or are entirely new.

6.1. Sigma Orientation Congruent Product Definition by the Sign Factor Function.

In this subsection we define the Clifford-like sigma orientation congruent algebra $\sigma\mathcal{OC}_{p,q}$ and provide formulas for computing it.

In the following subsection we demonstrate that the $\mathcal{OC}_{p,q}$ algebra of the primed GR axioms is a Clifford-like algebra. We accomplish this by proving that the $\mathcal{OC}_{p,q}$ algebra is identical to (or, more properly, isomorphic with) the explicitly Clifford-like $\sigma\mathcal{OC}_{p,q}$ algebra.[36]

In accordance with this fundamental definition we give an explicit formula for σ as a function of the two basis blades in the product. From this first formula for σ as a function of two basis blades we then derive a formula for σ as a function of the grades of any two homogeneous multivectors, but parametrized by the grade of the t-vector part of their Clifford product. In the end, by using the fundamental decomposition of the Clifford product, we obtain an explicit expression for the orientation congruent product of two arbitrary multivectors in terms of the sign factor function and their Clifford product. The proof of the keystone algebra isomorphism Theorem 6.9 is delayed until the next subsection.

[36] In another view we are proving the *deductive equivalence* of the primed orientation congruent axioms of the last section with that section's unprimed Clifford algebra axioms but having added to them as an axiom of existence Definition 6.5 for the (sigma) orientation congruent product.

In this section general set-theoretic sets as well as sets of basis vectors[37] are written as upper case letters in a calligraphic font such as $\mathcal{I}$. However, the power set function $\mathscr{P}$ as well as sets of blades or general multivectors are written in a script font. Also, $\#(\mathcal{I})$ denotes the *cardinality* of a set $\mathcal{I}$; $\mathscr{P}(\mathcal{I})$, the *power set* of $\mathcal{I}$; $\mathcal{I}^{\complement}$, the *set complement* of $\mathcal{I}$; and $\pm\mathcal{I} := \{\, a \mid \pm a \in \mathcal{I}\,\}$, the *negative extension* of $\mathcal{I}$. The symbol $\varnothing$ stands for the *empty set* $\{\ \}$.

First, we define notations for an ordered, orthonormal, set of basis vectors and various sets of basis blades derived from it. For $n = p+q$, let $\mathscr{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}, \mathbf{e}_n\}$ be an ordered set[38] of mutually orthogonal unit basis vectors for $\mathcal{OC}_{p,q}$ and its corresponding $\mathcal{C}\ell_{p,q}$. Then $\mathcal{B}\ell_{\mathscr{B}}$ signifies the set of basis blades for $\mathcal{OC}_{p,q}$ and $\mathcal{C}\ell_{p,q}$ generated from $\mathscr{B}$ by taking, for each subset of $\mathscr{B}$, the outer product[39] of all basis vectors in it in their prescribed order.[40] We use $\mathcal{B}\ell^{r}_{\mathscr{B}}$ to mean the set of basis blades generated by $\mathscr{B}$ which are of grade $2 \le r \le n$. We also make the definitions $\mathcal{B}\ell^{1}_{\mathscr{B}} := \mathscr{B}$ and $\mathcal{B}\ell^{0}_{\mathscr{B}} := 1$.

We also adopt the compact *multi-index notation* for the blades in the set $\pm\mathcal{B}\ell_{\mathscr{B}}$ (cf. Artin [4, pp. 186–188], Chevalley [44, p. 40], Deschamps [58, p. 688], Shaw [167, pp. 326 f.]). A multi-index is a sequence of integers from the set $\mathbb{Z}[1, n]$ that is written as a subscript to the base symbol for a basis vector. Generally, we use upper case italic letters for symbolic multi-indices as in $\mathbf{e}_I$. For a sequence $I = i_1, \ldots i_{r-1}, i_r$ of length $r \ge 2$ we define $\mathbf{e}_I := \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_{r-1}} \wedge \mathbf{e}_{i_r}$. The two limiting cases are treated as follows: if I is a sequence of length 1 with $I = i$, we define $\mathbf{e}_I := \mathbf{e}_i$; and if I is the empty sequence ε, we define $\mathbf{e}_\varepsilon := 1$. For example, $\mathbf{e}_{12} := \mathbf{e}_1 \wedge \mathbf{e}_2$. In this example, we have left out the separating comma in the sequence $1, 2$. Expressions following this convention will not be ambiguous as long as $n \le 9$. For more flexibility and to adapt to the natural basis of the orientation congruent algebra (used, for example, in Tables 5.6 and 5.10), we allow violation of the usual Clifford algebra convention that the sequence of integers in a multi-index must be ordered from least to greatest. Then, for example, we may write $\mathbf{e}_{21} = -\mathbf{e}_{12}$.

Next, we introduce a function $\operatorname{bset}(\bullet)$ which is implicitly parametrized by some ordered, orthonormal, set of basis vectors $\mathscr{B}$ for the orientation congruent algebra $\mathcal{OC}_{p,q}$ and its corresponding Clifford algebra $\mathcal{C}\ell_{p,q}$.

Definition 6.1 (Basis Set Function)**.**
We define the *basis set function* $\operatorname{bset}\colon \pm\mathcal{B}\ell_{\mathscr{B}} \to \mathscr{P}(\mathscr{B})$ such that for all $\mathbf{e}_I \in \pm\mathscr{B}^r$

$$\operatorname{bset}(\mathbf{e}_I) := \begin{cases} \{\, \mathbf{e}_{i_j} \in \mathscr{B} \mid \mathbf{e}_I = \pm\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_j} \wedge \cdots \wedge \mathbf{e}_{i_r} \,\}, & \text{if } r \ge 2; \\ \{\, \mathbf{e}_i \in \mathscr{B} \mid \mathbf{e}_I = \pm\mathbf{e}_i \,\}, & \text{if } r = 1; \\ \varnothing, & \text{if } r = 0. \end{cases}$$

[37] We except from this rule the set of *all* basis vectors for an algebra which we also write in a script font as $\mathscr{B}$.

[40] In this section $\mathscr{B}$ is *not* necessarily signature-ordered; that is, ordered such that all basis vectors of positive signature precede those of negative signature.

[40] Since the vectors in $\mathscr{B}$ are mutually orthogonal, $\mathbf{e}_i \wedge \mathbf{e}_j = \mathbf{e}_i \circ \mathbf{e}_j = \mathbf{e}_i \circledcirc \mathbf{e}_j$ for any $\mathbf{e}_i, \mathbf{e}_j \in \mathscr{B}$ (see [84], p. 15, Equation (89)). Therefore, $\mathcal{B}\ell_{\mathscr{B}} = \mathcal{CB}\ell_{\mathscr{B}} = \mathcal{OB}\ell_{\mathscr{B}}$ and any basis blade of $\mathcal{OC}_{p,q}$ is a basis blade of $\mathcal{C}\ell_{p,q}$. Also, we take the outer product of one factor to be that factor and the outer product of no factors to be the unit 1.

[40] Accordingly, the vectors in $\mathscr{B}$ are called the *generators* of $\mathcal{OC}_{p,q}$ (or $\mathcal{C}\ell_{p,q}$).

We call any set $\mathcal{I} \subseteq \mathscr{B}$ a *basis subset*. We also extend the function bset in the obvious way so that bset: $\mathscr{P}(\pm\mathcal{B}\ell_{\mathscr{B}}) \to \mathscr{P}(\mathscr{P}(\mathscr{B}))$,. Therefore, in particular, $\operatorname{bset}(\pm 1) = \varnothing$, $\operatorname{bset}(\pm\mathbf{e}_i) = \{\, \mathbf{e}_i \,\}$ for all $\mathbf{e}_i \in \mathscr{B}$, and $\operatorname{bset}(\pm\mathcal{B}\ell_{\mathscr{B}}) = \mathscr{P}(\mathscr{B})$.

In the following definition a special Clifford-like algebra is defined by changing the sign of the Clifford product $\circ$ of the Clifford algebra $\mathcal{C}\ell_n$ of a nondegenerate quadratic form given by our modified Hestenes-Sobczyk axioms. This definition of a special Clifford-like algebra is a slightly modified translation of the definition of a (*general*) Clifford-like algebra in the *maximally-graded* theory of Clifford algebras given by Hagmark and Lounesto ([83], [123, pp. 284 f.]). Hagmark and Lounesto define a Clifford-like algebra by specifying the sign of a certain product for basis blades that are indexed by binary n-tuples. Basis blades of this type belong to the theory of Clifford algebras defined as maximally-graded algebras. We obtain, instead, what I call a special Clifford-like algebra, if the Hagmark and Lounesto definition of a Clifford-like algebra is required to have an outer product equivalent to the one in our Definition 5.5. We use the special Clifford-like algebra below in our definition of the sigma orientation congruent algebra, rather than the (general) Clifford-like algebra that is exactly equivalent to the Clifford-like algebra of Hagmark and Lounesto, to avoid the possibility that the Clifford-like (outer) product of basis blades may differ by a sign from their Clifford (outer) product. Since the special Clifford-like algebras form a subset of the set of Clifford-like algebras and all the Clifford-like algebras we refer to in this paper are the special ones, in the rest of the paper following the next definition, we simply write *Clifford-like algebra* instead of *special Clifford-like algebra*.

Definition 6.2 ((Special) Clifford-Like Algebra)**.**
Let $\mathfrak{A}$ be an algebra, with the product, $\circledast_{\mathfrak{A}}$, that satisfies the first 24 axioms in Axiom Sets I through VI, Axiom VII.1′, Axiom VII.2 with a nondegenerate quadratic form $Q'_{p,q}$, $n = p + q$, that is not necessarily equal to the quadratic form Q°_n associated with $\mathcal{C}\ell_n$, and both bridge axioms in Axiom Set IX, with the $\mathfrak{A}$ algebra product $\circledast_{\mathfrak{A}}$ substituted for the Clifford product $\circ$ or the orientation congruent product $\circledcirc$ in all these axioms as is appropriate. Then $\mathfrak{A}$ is said to be a *(special) Clifford-like algebra* if and only if the $\mathfrak{A}$ product, $\circledast_{\mathfrak{A}}$, is the multilinear extension to all multivectors of the Clifford product of any two basis blades $\mathbf{e}_I$ and $\mathbf{e}_J$ modified by multiplying it by a *sign factor function*, $\sigma\colon \mathscr{P}(\mathscr{B}) \times \mathscr{P}(\mathscr{B}) \to \{\pm 1\}$, that is a function of the basis subsets $\mathcal{I} = \operatorname{bset}(\mathbf{e}_I)$ and $\mathcal{J} = \operatorname{bset}(\mathbf{e}_J)$ associated with $\mathbf{e}_I$ and $\mathbf{e}_J$:

$$\mathbf{e}_I \circledast_{\mathfrak{A}} \mathbf{e}_J = \sigma(\mathcal{I}, \mathcal{J})\, \mathbf{e}_I \circ \mathbf{e}_J. \tag{6.1}$$

Remark 6.3. With a suitable sign factor function we can define a Clifford-like algebra that is isomorphic to the Clifford algebra $\mathcal{C}\ell_{p,q}$ for any signature (p, q).

Remark 6.4. If we were to extend the image of the sign factor function to include 0, we could then give Clifford-like algebra definitions of all the common derived products of geometric algebra including the outer product and all types of inner products: Hestenes, fat dot, contractions, and scalar product.

Definition 6.5. We define the *sigma orientation congruent* algebra of a nondegenerate quadratic form $Q_{p,q}$, denoted by $\sigma\mathcal{OC}_{p,q}$, and with product denoted by a circled star[41] $\circledast$, as the Clifford-like algebra that is the multilinear extension to all multivectors of the multiplication rule

$$\mathbf{e}_I \circledast \mathbf{e}_J = \sigma(\mathcal{I}, \mathcal{J})\, \mathbf{e}_I \circ \mathbf{e}_J,^{42} \tag{6.2}$$

defined for all pairs of basis blades $\mathbf{e}_I, \mathbf{e}_J \in \mathcal{B}\ell_{\mathscr{B}}$, where the sign factor function of basis subsets σ is given by

$$\sigma(\mathcal{I}, \mathcal{J}) = (-1)^{\frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})[2\,\#(\mathcal{I})\,\#(\mathcal{J}) + \#(\mathcal{I}\cap\mathcal{J}) + 1]}, \tag{6.3}$$

for $\mathcal{I} = \mathrm{bset}(\mathbf{e}_I)$ and $\mathcal{J} = \mathrm{bset}(\mathbf{e}_J)$.

Now we begin to construct an explicit formula for the multilinear extension of Equations (6.2) and (6.3) for the sigma orientation congruent product $\circledast$ of $\sigma\mathcal{OC}_{p,q}$ given in Definition 6.5 to arbitrary multivectors.

The next lemma provides a formula for the sign factor function in terms of the two basis blade factors and the resultant basis blade of their Clifford product based on the relationship between the Clifford product of two basis blades and the symmetric difference of the sets of basis vectors "in" each of them.

Lemma 6.6. *For any $\mathbf{e}_I, \mathbf{e}_J \in \mathcal{B}\ell_{\mathscr{B}}$, if $\mathcal{I} = \mathrm{bset}(\mathbf{e}_I)$, $\mathcal{J} = \mathrm{bset}(\mathbf{e}_J)$, and $\mathcal{K} = \mathrm{bset}(\mathbf{e}_I \circ \mathbf{e}_J) = \mathrm{bset}(\mathbf{e}_I \circledast \mathbf{e}_J) \in \mathrm{bset}(\pm\mathcal{B}\ell_{\mathscr{B}})$, we may write the sign factor function σ of Definition 6.5 as*

$$\sigma(\mathbf{e}_I, \mathbf{e}_J) = (-1)^{\frac{1}{8}[\#(\mathcal{I})+\#(\mathcal{J})-\#(\mathcal{K})][4\,\#(\mathcal{I})\,\#(\mathcal{J})+\#(\mathcal{I})+\#(\mathcal{J})-\#(\mathcal{K})+2]}. \tag{6.4}$$

Proof. As is well known, if Δ denotes the *symmetric difference* operator on sets, and $\setminus$ denotes the *set difference* operator, then for all finite sets $\mathcal{I}$ and $\mathcal{J}$

$$\mathcal{I} \cap \mathcal{J} = (\mathcal{I} \cup \mathcal{J}) \setminus (\mathcal{I} \,\Delta\, \mathcal{J}) \quad \text{and} \tag{6.5a}$$

$$\#(\mathcal{I} \cap \mathcal{J}) = \frac{1}{2}[\#(\mathcal{I}) + \#(\mathcal{J}) - \#(\mathcal{I} \,\Delta\, \mathcal{J})]. \tag{6.5b}$$

Also for any $\mathbf{e}_I, \mathbf{e}_J \in \mathcal{B}\ell_{\mathscr{B}}$ we have

$$\mathrm{bset}(\mathbf{e}_I) \,\Delta\, \mathrm{bset}(\mathbf{e}_J) = \mathrm{bset}(\mathbf{e}_I \circ \mathbf{e}_J) = \mathrm{bset}(\mathbf{e}_I \circledast \mathbf{e}_J). \tag{6.6}$$

Then it is straightforward to rewrite Equation (6.3) of Definition 6.5 as Equation (6.4). □

In all the above we have had $\mathrm{bset}(\mathbf{e}_I \circ \mathbf{e}_J) = \mathrm{bset}(\mathbf{e}_I \circledast \mathbf{e}_J) \in \mathrm{bset}(\pm\mathcal{B}\ell_{\mathscr{B}}) = \mathscr{P}(\mathscr{B})$[43] for any $\mathbf{e}_I, \mathbf{e}_J \in \mathcal{B}\ell_{\mathscr{B}}$, thus ensuring that $\sigma(\mathbf{e}_I, \mathbf{e}_J)$ is well defined as a closed operation. Since generally the Clifford product of arbitrary (not necessarily basis) blades is no longer homogeneous, we may as well consider next the Clifford and orientation congruent products of homogeneous multivectors (which are not necessarily blades).

The form of Equation (6.4) for the sign factor function σ is suitable for generalization from products of basis blades to products of homogeneous multivectors. Simultaneously we parametrize σ by a grade index so that it is useful when $A \circ B$ is a general multivector rather than a blade in $\pm\mathcal{B}\ell_{\mathscr{B}}$. With these changes in the definition of the sign factor function of *basis blades* $\sigma(\mathbf{e}_I, \mathbf{e}_J)$ in Equation (6.3) of Definition 6.5 we obtain the definition of the sign factor function of the *grades of homogeneous multivectors* $\sigma_t(r, s)$ in Equation (6.8) of the next theorem.

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[42]Here we are using the symbol $\circledast$ for the product of the algebra $\sigma\mathcal{OC}_{p,q}$ at least until we prove that it is identical to the product $\circledcirc$ of the orientation congruent algebra $\mathcal{OC}_{p,q}$.

[42]Since $\mathbf{e}_I$ and $\mathbf{e}_J$ are basis blades, $\mathbf{e}_I \circ \mathbf{e}_J \in \pm\mathcal{B}\ell_{\mathscr{B}}$. In other words, $\pm\mathcal{B}\ell_{\mathscr{B}}$ is *closed* under any of the exterior, Clifford, or (sigma) orientation congruent products.

[43]Or, equivalently, $\mathrm{bset}(\mathbf{e}_I \circ \mathbf{e}_J) = \mathrm{bset}(\mathbf{e}_I \circledast \mathbf{e}_J) \subseteq \mathscr{B}$.

Theorem 6.7. *For any* homogeneous multivectors $A_r, B_s \in \sigma\mathcal{OC}_{p,q}$ *and* $\mathcal{C}\ell_{p,q}$, *with subscripts indicating their grades, the* multilinear extension *of the product* $\circledast$ *of the sigma orientation congruent algebra* $\sigma\mathcal{OC}_{p,q}$ *given by Definition 6.5 is*

$$A_r \circledast B_s = \sum_{t=|r-s|}^{r+s} \langle A_r \circledast B_s \rangle_t = \sum_{t=|r-s|}^{r+s} \sigma_t(r,s) \, \langle A_r \circ B_s \rangle_t, \tag{6.7}$$

where the sign factor function[44] $\sigma_t\colon \mathbb{Z}[0,n] \times \mathbb{Z}[0,n] \mapsto \pm\{\,1\,\}$, *now a function of the grades of* A_r, B_s *and parametrized by the grade* $t \in \mathbb{Z}[0,n]$ *of the* t*-vector part of* $A_r \circ B_s$, *is given by*

$$\sigma_t(r,s) = (-1)^{\frac{1}{8}[r+s-t][4rs+r+s-t+2]}. \tag{6.8}$$

Proof. The proof is immediate from Lemma 6.6 by the multilinearity of the Clifford product and the linearity of the grade selection operator. □

Using Equation (6.7) to evaluate the right hand side of the next equation we finally obtain an expression for the *sigma orientation congruent product of multivectors in terms of the sign factor function* $\sigma_t(r,s)$ and the Clifford product.

Corollary 6.8. *For all* $A, B \in \sigma\mathcal{OC}_{p,q}$

$$A \circledast B = \sum_{r,s} \langle A \rangle_r \circledast \langle B \rangle_s \text{ as evaluated by Equation (6.7)}. \tag{6.9}$$

Proof. The proof is immediate from Lemma 6.6 by the multilinearity of the Clifford product and the linearity of the grade selection operator. □

6.2. **Equivalence of the $\boldsymbol{\sigma}$ and the HS Orientation Congruent Algebras.** We next prove Theorem 6.9. This fundamental isomorphism theorem states that the orientation congruent product of Corollary 6.8, derived from the sign factor function and the fundamental decomposition of the Clifford product, and the orientation congruent product, defined by the modified Axiom Sets I′ through VII′, and the new Axioms Set VIII′, are equivalent. Our theorem and proof closely follows a similar theorem of Lounesto and his proof [123, pp. 282 f.].

In the following proof, as is allowed, we restrict the factors in all products to be basis blades in $\mathcal{B}\ell_{\mathscr{B}}$. So from another viewpoint we are directly proving an implicit keystone theorem that the formula for the sign factor function given by Equation (6.3) in Definition 6.5 is correct. This equation is the foundation from which all of Lemma 6.6, Theorem 6.7, Corollary 6.8, and the Fundamental $\mathcal{OC}$ Product Decomposition Theorem (Theorem 7.2 of Section 7) follow.

Let $\mathscr{B} = \{\,\mathbf{e}_1, \ldots, \mathbf{e}_p, \mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+q}\,\}$ be an arbitrary signature-ordered, orthonormal, set of basis vectors for $V^n \subseteq \mathcal{CO}_{p,q}$. In other words, for all $\mathbf{e}_i, \mathbf{e}_j \in \mathscr{B}$ and all integers $1 \le i, j \le n$, where $n = p + q$,

$$\begin{aligned} {\mathbf{e}_i}^2 = \mathbf{e}_i \circledcirc \mathbf{e}_i = \mathbf{e}_i \circ \mathbf{e}_i &= \begin{cases} +1, & \text{if } \quad 1 \le i \le p, \quad \text{and} \\ -1, & \text{if } p+1 \le i \le p+q = n, \text{ and} \end{cases} \\ \mathbf{e}_i \circledcirc \mathbf{e}_j = \mathbf{e}_i \circ \mathbf{e}_j &= -\mathbf{e}_j \circledcirc \mathbf{e}_i = -\mathbf{e}_j \circ \mathbf{e}_i, \quad \text{if } i \ne j. \end{aligned} \tag{6.10}$$

The Fundamental $\boldsymbol{\sigma\mathcal{OC}}$-$\boldsymbol{\mathcal{OC}}$ Algebra Isomorphism Theorem

[44] Here we are about to use the convenient notation $\mathbb{Z}[a,b] := \{\, i \mid i \in \mathbb{Z} \text{ and } a \le i \le b \,\}$.

Theorem 6.9. *The Clifford-like algebra $\sigma\mathcal{OC}_{p,q}$ that is the multilinear extension to all multivectors of the multiplication rule*

$$\mathbf{e}_I \circledast \mathbf{e}_J = \sigma(\mathbf{e}_I, \mathbf{e}_J)\, \mathbf{e}_I \circ \mathbf{e}_J,$$

between all pairs of basis blades $\mathbf{e}_I, \mathbf{e}_J \in \mathcal{B}\ell_{\mathscr{B}}$, where σ is defined by Equation (6.3) *of Definition 6.5, is identical to the orientation congruent algebra $\mathcal{OC}_{p,q}$ with product $\circledcirc$ defined by the modified (primed) Axiom Sets I′ through VII′, and the new (unprimed) Axiom Sets VIII and IX , along with the modified (primed) Definitions 5.4′ through 5.9′, and the new (unprimed) Definitions 5.10 and 5.11.*

Proof. The proof of Theorem 6.9 consists of two main parts:

(1) the proof that sigma orientation congruent product of Definition 6.5 satisfies the primed Hestenes-Sobczyk axioms for the orientation congruent algebra given in section 5, and
(2) the proof that the Hestenes-Sobczyk axioms for the orientation congruent algebra given in section 5 determine the formula for sign factor function given by Equation (6.3) of Definition 6.5.

6.2.1. *Proof that the Sigma Orientation Congruent Product Satisfies the Orientation Congruent Algebra's Hestenes-Sobczyk Axioms.* Actually, we explicitly prove (1) for only the last four axioms in Axiom Sets VII′ and VIII: Axioms VII.1′, VII.2′, VIII.1, and VIII.2. A proof for these four alone is sufficient because they are the only axioms that are either material modifications of some unprimed axiom or are entirely new.

It is sufficient to show that $\sigma\mathcal{OC}_{p,q}$ is generated by n anticommuting vectors with squares of ± 1 given by Equation (6.10), has a unit element, and that the blades in $\mathcal{B}\ell_{\mathscr{B}}$ satisfy Axioms VII.1′, VII.2′, and VIII.2 under the $\circledast$ product of $\sigma\mathcal{OC}_{p,q}$.

Consider the first requirement. Since the product of $\sigma\mathcal{OC}_{p,q}$ is simply the Clifford product multiplied by a sign factor of ± 1, it has the same set of generators $\mathscr{B}$ as the Clifford algebra $\mathcal{C}\ell_{p,q}$. Thus this requirement is fulfilled.

Next consider the second requirement. By definition, for any $\mathbf{e}_I \in \mathcal{B}\ell_{\mathscr{B}}$, $1 \circledast \mathbf{e}_I = \sigma(1, \mathbf{e}_I)\, 1 \circ \mathbf{e}_I$. But

$$\begin{aligned}\sigma(1, \mathbf{e}_I) &= (-1)^{\frac{1}{2}\,\#(\varnothing \cap \mathrm{bset}(\mathbf{e}_I))[2\,\#(\varnothing)\,\#(\mathrm{bset}(\mathbf{e}_I)) + \#(\varnothing \cap \mathrm{bset}(\mathbf{e}_I)) + 1]} \\ &= (-1)^0 = 1,\end{aligned}$$

Inspection of Equation (6.3) shows that σ is symmetric in its arguments. Thus, $1 \circledast \mathbf{e}_I = 1 \circ \mathbf{e}_I = \mathbf{e}_I$ and both multiplications commute. Therefore, as required, the unit of algebra $\sigma\mathcal{OC}_{p,q}$ exists; it is the scalar 1.

Proof for Axiom VII.1′

We recall that Axiom VII.1′ requires that the outer product of multivectors is associative: For all $A, B, C \in \mathcal{OC}_{p,q}$ or $\mathcal{C}\ell_{p,q}$

$$(A \wedge B) \wedge C = A \wedge (B \wedge C).$$

Restricting A, B, and C to be homogeneous, with subscripts indicating their grades, we obtain this equation:

$$(A_r \wedge B_s) \wedge C_t = A_r \wedge (B_s \wedge C_t).$$

Substituting the basis blades $\mathbf{e}_I \in \mathscr{B}^r$, $\mathbf{e}_J \in \mathscr{B}^s$, and $\mathbf{e}_K \in \mathscr{B}^t$ we have

$$(\mathbf{e}_I \wedge \mathbf{e}_J) \wedge \mathbf{e}_K = \mathbf{e}_I \wedge (\mathbf{e}_J \wedge \mathbf{e}_K). \tag{6.11}$$

Using Equation (5.2) of Definition 5.5 for the outer product we may write

$$\langle\langle \mathbf{e}_I \circledast \mathbf{e}_J\rangle_{i+j} \circledast \mathbf{e}_K\rangle_{i+j+k} = \langle \mathbf{e}_I \circledast \langle \mathbf{e}_J \circledast \mathbf{e}_K\rangle_{j+k}\rangle_{i+j+k}. \tag{6.12}$$

Applying Equation (6.2) of Definition 6.5 gives

$$\begin{aligned}\sigma(\mathbf{e}_I, \mathbf{e}_J)\ \sigma(\mathbf{e}_I \circ \mathbf{e}_J, \mathbf{e}_K)\, \langle\langle \mathbf{e}_I \circ \mathbf{e}_J\rangle_{r+s} \circ \mathbf{e}_K\rangle_{r+s+t} &= \\ \sigma(\mathbf{e}_J, \mathbf{e}_K)\ \sigma(\mathbf{e}_J \circ \mathbf{e}_K, \mathbf{e}_I)\, &\langle \mathbf{e}_I \circ \langle \mathbf{e}_J \circ \mathbf{e}_K\rangle_{s+t}\rangle_{r+s+t}.\end{aligned} \tag{6.13}$$

Now we let $\mathcal{I} = \text{bset}(\mathbf{e}_I)$, $\mathcal{J} = \text{bset}(\mathbf{e}_J)$, and $\mathcal{K} = \text{bset}(\mathbf{e}_K)$ and use Equation (6.3) to perform the next two evaluations. On the left hand side, evaluating the sign factor functions gives

$$\begin{aligned}\sigma(\mathbf{e}_I, \mathbf{e}_J) \cdot \sigma(\mathbf{e}_I \circ \mathbf{e}_J, \mathbf{e}_K) &= (-1)^{\frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})[2\,\#(\mathcal{I})\,\#(\mathcal{J})+\#(\mathcal{I}\cap\mathcal{J})+1]} \\ &\cdot (-1)^{\frac{1}{2}\,\#((\mathcal{I}\Delta\mathcal{J})\cap\mathcal{K})[2\,\#(\mathcal{I}\Delta\mathcal{J})\,\#(\mathcal{K})+\#((\mathcal{I}\Delta\mathcal{J})\cap\mathcal{K})+1]}.\end{aligned} \tag{6.14}$$

On the right hand side, evaluating the sign factor functions gives

$$\begin{aligned}\sigma(\mathbf{e}_J, \mathbf{e}_K) \cdot \sigma(\mathbf{e}_J \circ \mathbf{e}_K, \mathbf{e}_I) &= (-1)^{\frac{1}{2}\,\#(\mathcal{J}\cap\mathcal{K})[2\,\#(\mathcal{J})\,\#(\mathcal{K})+\#(\mathcal{J}\cap\mathcal{K})+1]} \\ &\cdot (-1)^{\frac{1}{2}\,\#((\mathcal{J}\Delta\mathcal{K})\cap\mathcal{I})[2\,\#(\mathcal{J}\Delta\mathcal{K})\,\#(\mathcal{I})+\#((\mathcal{J}\Delta\mathcal{K})\cap\mathcal{I})+1]}.\end{aligned} \tag{6.15}$$

Using Equations (6.5b) and (6.6) we observe that if (and only if) at least one of $\mathcal{I} \cap \mathcal{J}$, $\mathcal{I} \cap \mathcal{J}$, or $\mathcal{I} \cap \mathcal{J}$ is nonempty, both sides of Equation (6.13) are 0. In this case the values of the sign factor functions are irrelevant.

If $\mathcal{I} \cap \mathcal{J}$, $\mathcal{J} \cap \mathcal{K}$, and $\mathcal{I} \cap \mathcal{K}$ are all equal to $\varnothing$, both sides of Equation (6.13) are nonzero and the $(r+s+t)$-grade part of the Clifford products on the left hand side of Equation (6.13) is equal to that on the right hand side.[45] In this case the question of equality in Equation (6.13) hinges only on the values of the sign factor functions.

Examining the right hand sides of both Equations (6.14) and (6.15), we see that the first "cardinality factors," $\#(\mathcal{I} \cap \mathcal{J})$ and $\#(\mathcal{J} \cap \mathcal{K})$, in the exponent of the first -1 are obviously 0 when $\mathcal{I} \cap \mathcal{J} = \mathcal{J} \cap \mathcal{K} = \mathcal{I} \cap \mathcal{K} = \varnothing$. Thus, this first -1 raised to the power zero becomes 1 in both Equation (6.14) and (6.15).

Consider now the first cardinality factor of the second -1 on the right hand side of Equation (6.14); it is $\#((\mathcal{I}\,\Delta\,\mathcal{J}) \cap \mathcal{K})$. We perform some elementary set-theoretic manipulations[46] on $(\mathcal{I}\,\Delta\,\mathcal{J}) \cap \mathcal{K}$:

$$\begin{aligned}(\mathcal{I}\,\Delta\,\mathcal{J}) \cap \mathcal{K} &= [(\mathcal{I} \cup \mathcal{J}) \setminus (\mathcal{I} \cap \mathcal{J})] \cap \mathcal{K} \\ &= (\mathcal{I} \cup \mathcal{J}) \cap (\mathcal{I} \cap \mathcal{J})^{\complement} \cap \mathcal{K} \\ &= [(\mathcal{I} \cup \mathcal{J}) \cap \mathcal{K}] \cap (\mathcal{I} \cap \mathcal{J})^{\complement} \\ &= [(\mathcal{I} \cap \mathcal{K}) \cup (\mathcal{J} \cap \mathcal{K})] \cap (\mathcal{I} \cap \mathcal{J})^{\complement}.\end{aligned}$$

Since $\mathcal{I} \cap \mathcal{K} = \mathcal{J} \cap \mathcal{K} = \varnothing$, by the expression in the last line above we find that $(\mathcal{I}\,\Delta\,\mathcal{J}) \cap \mathcal{K} = \varnothing$. Therefore the first cardinality factor of the second -1 on the right hand side of Equation (6.14) is 0. This makes that exponentiated -1 become

[45]For a proof see Reference [84, p. 11, Eq. (57)].

[46]We use the notation $\mathcal{I}^{\complement}$ for the *set complement* of $\mathcal{I}$ with respect to bset $(\mathbf{\Omega}) = \{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$ as the universal set.

1. Similar manipulations lead to the same conclusion for Equation (6.15). Thus, the sign factor functions are all unity and we have proved the equality of both sides of Equation (6.13). This, in turn, implies that Equation (6.11) is true. Therefore, Axiom VII.1′, restricted to the set of basis blades $\mathcal{B}\ell_{\mathscr{B}}$, is satisfied under the product $\circledast$ of the sigma orientation congruent algebra $\sigma\mathcal{OC}_{p,q}$.

Proof for Axiom VII.2′

Axiom VII.2′ requires that the square of an r-blade and the product of the quadratic forms of the vectors in it be equal. So we restrict formula (6.3) for the sign factor function σ to two identical basis blades $\mathbf{e}_i, \mathbf{e}_i \in \mathscr{B}$ and apply the set-theoretic identity $\mathcal{I} \cap \mathcal{I} = \mathcal{I}$ to get

$$\sigma(\mathbf{e}_I, \mathbf{e}_I) = (-1)^{\frac{1}{2}\,\#(\mathrm{bset}(\mathbf{e}_I))[2\,\#(\mathrm{bset}(\mathbf{e}_I))\,\#(\mathrm{bset}(\mathbf{e}_I))+\#(\mathrm{bset}(\mathbf{e}_I))+1]}.$$

Since $\#(\mathrm{bset}(\mathbf{e}_I)) = r$, we obtain

$$\sigma(\mathbf{e}_I, \mathbf{e}_I) = (-1)^{\frac{1}{2}r(2r^2+r+1)}.$$

Since $r(2r^2 + r + 1) \equiv r(r-1) \mod 4$, we have

$$\sigma(\mathbf{e}_I, \mathbf{e}_I) = (-1)^{\frac{1}{2}r(r-1)}.$$

Therefore, $\mathbf{e}_I \circledast \mathbf{e}_I = (-1)^{\frac{1}{2}r(r-1)}\mathbf{e}_I \circ \mathbf{e}_I$. However, $(-1)^{\frac{1}{2}r(r-1)}\mathbf{e}_I$ is just the usual formula for ${\mathbf{e}_I}^{\dagger}$ the reversion of $\mathbf{e}_I$. Thus, we obtain

$$\begin{aligned}\mathbf{e}_I \circledast \mathbf{e}_I &= {\mathbf{e}_I}^{\dagger} \circ \mathbf{e}_I \\ &= Q(\mathbf{e}_{i_1}) \cdots Q(\mathbf{e}_{i_j}) \cdots Q(\mathbf{e}_{i_r}).\end{aligned}$$

Here we have let $\mathbf{e}_I = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_j} \wedge \cdots \wedge \mathbf{e}_{i_r}$ with all $\mathbf{e}_{i_j} \in \mathscr{B}$ basis vectors. Therefore, Axiom VII.2′, restricted to the set of basis blades $\mathcal{B}\ell_{\mathscr{B}}$, is satisfied under the product $\circledast$ of the sigma orientation congruent algebra $\sigma\mathcal{OC}_{p,q}$.

Proof for Axiom VIII.2

Axiom VIII.2 requires that within $\mathcal{OC}_{p,q}$, or its arbitrary extension by one dimension, for all nonempty subsets $\mathscr{A}$ of multivectors there exists a (nonunique) nonscalar, unit magnitude blade $\boldsymbol{\omega}_{\mathscr{A}}$, called the counit of $\mathscr{A}$, which has the generalized commutativity of right $\boldsymbol{\omega}_{\mathscr{A}}$-complementation property for all multivectors in $\mathscr{A}$. In addition, Axiom VIII.2 states that an extension of $\mathcal{OC}_{p,q}$ by one dimension always exists. So in the following proof the symbol $\mathscr{B}$ for the basis set will refer to either the original basis or its extension to $\mathscr{B} \cup \{\, \mathbf{e}_{n+1} \,\}$, if necessary.[47] Of course, the meaning of $\mathcal{B}\ell_{\mathscr{B}}$ must also be modified to reflect any change made to that of $\mathscr{B}$.

As is sufficient for the proof, we restrict $\mathscr{A}$ to be $\varnothing \subset \mathscr{A} \subseteq \mathcal{B}\ell_{\mathscr{B}}$. We claim that any basis blade $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \in \mathscr{B}^r$ such that $r = \#(\mathrm{bset}(\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}))$ is odd and

$$\mathrm{bset}(\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}) = \mathcal{B} \cup \bigcup_{\mathbf{e}_I \in \mathscr{A}} \mathrm{bset}(\mathbf{e}_I) \quad \text{for some } \varnothing \subseteq \mathcal{B} \subseteq \mathscr{B} \tag{6.16}$$

satisfies these requirements.

Before continuing we must first show that a set $\mathcal{B}$ that makes $r = \#(\mathrm{bset}(\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}))$ odd always exists. If the union of the $\mathrm{bset}(\mathbf{e}_I)$ over $\mathscr{A}$ is already of odd cardinality,

[47] Here the subscript $n+1$ is not intended to imply that $Q(\mathbf{e}_{n+1})$ is necessarily negative and neither is the symbol Q meant to imply that $Q = Q_{p,q+1}$ rather than $Q = Q_{p+1,q}$.

we may choose $\mathcal{B} = \varnothing$. If it is of even cardinality, we may choose $\mathcal{B} = \{\mathbf{e}_j\}$ where $\mathbf{e}_j$ is any basis vector not in the union of the bset$(\mathbf{e}_I)$ over $\mathscr{A}$. In this last case we may have to choose $\mathcal{B} = \{\mathbf{e}_{n+1}\}$, where $\mathbf{e}_{n+1}$ is the basis vector used to extend the dimension of the original vector space V^n to the odd number $n+1$.

Next, preliminary to the heart of the proof, we show the alleged counit $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}$ satisfies the simpler requirements of Axiom VIII.2. Since we have managed to make $r = \#(\text{bset}(\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}))$ odd by a proper choice of $\mathcal{B}$, the condition that $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}$ must not be a scalar is satisfied. Also, the requirement that $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}$ must have unit magnitude holds because $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}$ is a member of the set of basis blades $\mathcal{B}\ell_{\mathscr{B}}$ generated by the orthonormal basis $\mathscr{B}$.

Next, as a lemma to the main proof that the sign factor function satisfies Axiom VIII.2, we prove that it satisfies Theorem 5.12. This theorem states that, under the same conditions as for Axiom VIII.2, $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}$ commutes with all multivectors in $\mathscr{A}$.

For Theorem 5.12 let $\mathbf{e}_J \in \mathcal{B}\ell_{\mathscr{B}}$ be some basis blade in $\mathscr{A}$. Also, let s be the grade of $\mathbf{e}_J$. Then

$$\begin{aligned} \mathbf{e}_J \circledast \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} &= \sigma(\mathbf{e}_J, \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}})\, \mathbf{e}_J \circ \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \quad \text{and} \\ \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \circledast \mathbf{e}_J &= \sigma(\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}, \mathbf{e}_J)\, \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \circ \mathbf{e}_J. \end{aligned}$$

As previously observed σ is patently commutative in its arguments. Therefore we must show that $\mathbf{e}_J \circ \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} = \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \circ \mathbf{e}_J$.

At this point it is convenient to appeal to formulas involving the so-called *fat dot* or *modified Hestenes inner product* [16, 61], [122, pp. 143 f.] of the Clifford algebra $\mathcal{C}\ell_{p,q}$. So we first define this new inner product, then we provide an equation relating it to the Clifford product.

The fat dot inner product is almost the same as the (unmodified) Hestenes inner product. The only difference is that the Hestenes inner product is restricted to a zero result when either operand is a scalar [84, p. 6, Eq. (18)], but the fat dot inner product is not. Thus we have the following definition.

Definition 6.10. For all general multivectors $A, B \in \mathcal{C}\ell_{p,q}$, the *fat dot inner product*, written with a large centered dot $\bullet$, is defined as

$$A \bullet B := \sum_{r,s} \langle \langle A\rangle_r \circ \langle B\rangle_s \rangle_{|r-s|}. \tag{6.17}$$

The next theorem relates the fat dot inner product to the Clifford product.

Theorem 6.11. *Let $\mathbf{A}_r, \mathbf{B}_s \in \mathcal{C}\ell_{p,q}$ be blades written with subscripts indicating their grades. Then by Definition 5.4 each can be written as an orientation congruent multiproduct, with any grouping into binary products, of r or s pairwise anticommuting vectors. In particular, let $\mathbf{B}_s = \mathbf{b}_1 \circledcirc \cdots \circledcirc \mathbf{b}_i \circledcirc \cdots \circledcirc \mathbf{b}_s$ where all $\mathbf{a}_i \in V^n$ and $\mathbf{a}_i \circledcirc \mathbf{a}_j = -\mathbf{a}_j \circledcirc \mathbf{a}_i$ for all $i \neq j$.*

with $\mathbf{A}_r \wedge \mathbf{b}_i = 0$ for all $1 \le i \le s$. Therefore $r \ge s$ and

$$\begin{aligned} \mathbf{B}_s \bullet \mathbf{A}_r &= \mathbf{B}_s \circ \mathbf{A}_r \quad \textit{and} \\ \mathbf{A}_r \bullet \mathbf{B}_s &= \mathbf{A}_r \circ \mathbf{B}_s. \end{aligned} \tag{6.18}$$

Proof. The proof follows from, Theorem 7.1, the Fundamental Clifford Product Decomposition Theorem and Definition. □

By Equation (6.16), $\varnothing \subset \operatorname{bset}(\mathbf{e}_J) \subseteq \operatorname{bset}(\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}})$. Therefore,

$$\begin{aligned} \mathbf{e}_J \circ \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} &= \mathbf{e}_J \bullet \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \\ \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \circ \mathbf{e}_J &= \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \bullet \mathbf{e}_J, \end{aligned}$$

where the large centered dot $\bullet$ denotes the so-called *fat dot* or *modified Hestenes inner product* of the Clifford algebra $\mathcal{C}\ell_{p,q}$ [16, 61], [122, pp. 143 f.].

Consider next the rule for commuting the operands of the Hestenes inner product given by Harke's paper [84, p. 6, Eq. (22)]. For homogeneous multivectors with subscripts indicating their grades

$$A_r \cdot B_s = (-1)^{s(s-r)} B_s \cdot A_r, \text{ if } r \geq s, \tag{6.19}$$

where the small centered dot $\cdot$ denotes the Hestenes inner product. It is easily seen that Harke's derivation of Equation (6.19) also applies to the fat dot inner product. Therefore we may write

$$A_r \bullet B_s = (-1)^{s(s-r)} B_s \bullet A_r, \text{ if } r \geq s, \tag{6.20}$$

which is Equation (6.19) with the fat dot inner product substituted for the Hestenes inner product.

Finally, let $A_r = \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}$ and $B_s = \mathbf{e}_J$. Applying Equation (6.20) above for commutation of the fat dot inner product, we see that

$$\mathbf{e}_J \circ \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} = \mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} \circ \mathbf{e}_J,$$

since $r \geq s$ by the defining Equation (6.16) for $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}$ and $r = \#(\operatorname{bset}(\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}))$ is odd by assumption. Therefore, Theorem 5.12, restricted to the set of basis blades $\mathcal{B}\ell_{\mathscr{B}}$, is satisfied under the product $\circledast$ of the sigma orientation congruent algebra $\sigma\mathcal{OC}_{p,q}$.

Now we present the main part of the proof. Axiom VIII.2 requires that the right $\boldsymbol{\omega}_{\mathscr{A}}$-complement commutes over the two factors and the result of an orientation congruent product of multivectors: For all $\mathscr{A}$ that are nonempty sets of multivectors, $\varnothing \subset \mathscr{A} \subseteq \mathcal{OC}_{p,q}$, all counits $\boldsymbol{\omega}_{\mathscr{A}}$ of $\mathscr{A}$, and all $A, B \in \mathscr{A}$

$$A^{\boldsymbol{\omega}_{\mathscr{A}}} \circledcirc B = A \circledcirc B^{\boldsymbol{\omega}_{\mathscr{A}}} = (A \circledcirc B)^{\boldsymbol{\omega}_{\mathscr{A}}}.$$

It is again sufficient to restrict $\mathscr{A}$ to a set of basis blades $\varnothing \subset \mathscr{A} \subseteq \mathcal{B}\ell_{\mathscr{B}}$. Then the general multivectors A, B become the basis blades $\mathbf{e}_I, \mathbf{e}_J \in \mathscr{A}$. Let $\mathbf{e}_I \in \mathscr{B}^r$ and $\mathbf{e}_J \in \mathscr{B}^s$. Let the counit $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}}$ defined by Equation (6.16) be $\mathbf{e}_K \in \mathscr{B}^t$. Then, substituting the $\sigma\mathcal{OC}_{p,q}$ algebra product $\circledast$ for the orientation congruent algebra product $\circledcirc$ in the last equation, we obtain

$$\mathbf{e}_I{}^{\mathbf{e}_K} \circledast \mathbf{e}_J = \mathbf{e}_I \circledast \mathbf{e}_J{}^{\mathbf{e}_K} = (\mathbf{e}_I \circledast \mathbf{e}_J)^{\mathbf{e}_K}.$$

By lowering the superscript $\mathbf{e}_K$ in the compact $\boldsymbol{\omega}_{\mathscr{A}}$-complementation notation of the last equation we arrive at

$$(\mathbf{e}_I \circledast \mathbf{e}_K) \circledast \mathbf{e}_J = \mathbf{e}_I \circledast (\mathbf{e}_J \circledast \mathbf{e}_K) = (\mathbf{e}_I \circledast \mathbf{e}_J) \circledast \mathbf{e}_K.$$

Applying Equation (6.2) of Definition 6.5 to this last result gives the following three equal expressions:

$$\sigma(\mathbf{e}_I, \mathbf{e}_K) \cdot \sigma(\mathbf{e}_I \circ \mathbf{e}_K, \mathbf{e}_J)\, (\mathbf{e}_I \circ \mathbf{e}_K) \circ \mathbf{e}_J, \tag{6.21a}$$

$$\sigma(\mathbf{e}_J, \mathbf{e}_K) \cdot \sigma(\mathbf{e}_J \circ \mathbf{e}_K, \mathbf{e}_I)\;\; \mathbf{e}_I \circ (\mathbf{e}_J \circ \mathbf{e}_K), \text{ and} \tag{6.21b}$$

$$\sigma(\mathbf{e}_I, \mathbf{e}_J) \cdot \sigma(\mathbf{e}_I \circ \mathbf{e}_J, \mathbf{e}_K)\, (\mathbf{e}_I \circ \mathbf{e}_J) \circ \mathbf{e}_K. \tag{6.21c}$$

The double Clifford products on the right side of all three expressions in Equation (6.21) are equal because the Clifford product is associative and because, as we have already proved, the counit $\mathbf{e}_{\boldsymbol{\omega}_{\mathscr{A}}} = \mathbf{e}_K$ commutes with all multivectors in its "generating" set $\mathscr{A}$. So next we look at the sign factor functions in these expressions.

Let $\mathcal{I} = \mathrm{bset}(\mathbf{e}_I)$, $\mathcal{J} = \mathrm{bset}(\mathbf{e}_J)$, and $\mathcal{K} = \mathrm{bset}(\mathbf{e}_K)$. Then starting with Equation (6.3) of Definition 6.5 we evaluate and simplify the sign factor functions of expressions (6.21a) and (6.21c), each in turn, until we obtain two equivalent expressions. The analogous manipulation of expression (6.21b) corresponding to what we do to (6.21a) is left to the curious reader.

Evaluating the sign factor functions of expression (6.21a) gives

$$\begin{aligned}\sigma(\mathbf{e}_I, \mathbf{e}_K) \cdot \sigma(\mathbf{e}_I \circ \mathbf{e}_K, \mathbf{e}_J) = (-1)^{\frac{1}{2}\#(\mathcal{I}\cap\mathcal{K})[2\#(\mathcal{I})\#(\mathcal{K})+\#(\mathcal{I}\cap\mathcal{K})+1]} \\ \cdot (-1)^{\frac{1}{2}\#((\mathcal{I}\Delta\mathcal{K})\cap\mathcal{J})[2\#(\mathcal{I}\Delta\mathcal{K})\#(\mathcal{J})+\#((\mathcal{I}\Delta\mathcal{K})\cap\mathcal{J})+1]}.\end{aligned}$$

Using set-theoretic identities[48] to simplify the above expression gives

$$\begin{aligned}\sigma(\mathbf{e}_I, \mathbf{e}_K) \cdot \sigma(\mathbf{e}_I \circ \mathbf{e}_K, \mathbf{e}_J) = (-1)^{\frac{1}{2}\#(\mathcal{I})[2\#(\mathcal{I})\#(\mathcal{K})+\#(\mathcal{I})+1]} \\ \cdot (-1)^{\frac{1}{2}\#\left(\mathcal{J}\cap\mathcal{I}^{\complement}\right)\left[2\#\left(\mathcal{K}\cap\mathcal{I}^{\complement}\right)\#(\mathcal{J})+\#\left(\mathcal{J}\cap\mathcal{I}^{\complement}\right)+1\right]}.\end{aligned}$$

Recall the identity $\#\left(\mathcal{J} \cap \mathcal{I}^{\complement}\right) = \#(\mathcal{J}) - \#(\mathcal{I} \cap \mathcal{J})$; also, $\#\left(\mathcal{K} \cap \mathcal{I}^{\complement}\right) = \#(\mathcal{K}) - \#(\mathcal{I})$, since $\mathcal{I} \subseteq \mathcal{K}$. Substituting these in the last expression produces

$$\begin{aligned}\sigma(\mathbf{e}_I, \mathbf{e}_K) \cdot \sigma(\mathbf{e}_I \circ \mathbf{e}_K, \mathbf{e}_J) = (-1)^{\frac{1}{2}\#(\mathcal{I})[2\#(\mathcal{I})\#(\mathcal{K})+\#(\mathcal{I})+1]} \\ \cdot (-1)^{\frac{1}{2}[\#(\mathcal{J})-\#(\mathcal{I}\cap\mathcal{J})]\cdot[2\{\#(C)-\#(\mathcal{I})\}\#(\mathcal{J})+\#(\mathcal{J})-\#(\mathcal{I}\cap\mathcal{J})+1]}.\end{aligned}$$

Since $\#(\mathcal{K})$ must always be odd, we may further simplify to

$$\begin{aligned}\sigma(\mathbf{e}_I, \mathbf{e}_K) \cdot \sigma(\mathbf{e}_I \circ \mathbf{e}_K, \mathbf{e}_J) = (-1)^{\frac{1}{2}\#(\mathcal{I})[2\#(\mathcal{I})+\#(\mathcal{I})+1]} \\ \cdot (-1)^{\frac{1}{2}[\#(\mathcal{J})-\#(\mathcal{I}\cap\mathcal{J})]\cdot[2\{1-\#(\mathcal{I})\}\#(\mathcal{J})+\#(\mathcal{J})-\#(\mathcal{I}\cap\mathcal{J})+1]}.\end{aligned}$$

Multiplying out the exponents and simplifying them mod 2 gives

$$\begin{aligned}\sigma(\mathbf{e}_I, \mathbf{e}_K) \cdot \sigma(\mathbf{e}_I \circ \mathbf{e}_K, \mathbf{e}_J) = (-1)^{[\#(\mathcal{I})+\#(\mathcal{J})+\#(\mathcal{I})\#(\mathcal{J})+\#(\mathcal{I})\#(\mathcal{J})\#(\mathcal{I}\cap\mathcal{J})]} \\ \cdot (-1)^{\frac{1}{2}\#(\mathcal{I})[\#(\mathcal{I})+1]} \cdot (-1)^{\frac{1}{2}\#(\mathcal{J})[\#(\mathcal{J})+1]} \cdot (-1)^{\frac{1}{2}\#(\mathcal{I}\cap\mathcal{J})[\#(\mathcal{I}\cap\mathcal{J})-1]}.\end{aligned} \tag{6.22}$$

We now shift attention to expression (6.21c) whose sign factor functions evaluate to give

$$\begin{aligned}\sigma(\mathbf{e}_I, \mathbf{e}_J) \cdot \sigma(\mathbf{e}_I \circ \mathbf{e}_J, \mathbf{e}_K) = (-1)^{\frac{1}{2}\#(\mathcal{I}\cap\mathcal{J})[2\#(\mathcal{I})\#(\mathcal{J})+\#(\mathcal{I}\cap\mathcal{J})+1]} \\ \cdot (-1)^{\frac{1}{2}\#((\mathcal{I}\Delta\mathcal{J})\cap\mathcal{K})[2\#(\mathcal{I}\Delta\mathcal{J})\#(\mathcal{K})+\#((\mathcal{I}\Delta\mathcal{J})\cap\mathcal{K})+1]}.\end{aligned}$$

Removing the odd factor $\#(\mathcal{K})$, multiplying out and separating certain exponents, replacing $\#((\mathcal{I} \,\Delta\, \mathcal{J}) \cap \mathcal{K})$ with $\#(\mathcal{I} \,\Delta\, \mathcal{J})$, and applying mod 2 identities yields

$$\begin{aligned}\sigma(\mathbf{e}_I, \mathbf{e}_J) \cdot \sigma(\mathbf{e}_I \circ \mathbf{e}_J, \mathbf{e}_K) = (-1)^{[\#(\mathcal{I}\Delta\mathcal{J})+\#(\mathcal{I})\#(\mathcal{J})\#(\mathcal{I}\cap\mathcal{J})]} \\ \cdot (-1)^{\frac{1}{2}\#(\mathcal{I}\cap\mathcal{J})[\#(\mathcal{I}\cap\mathcal{J})+1]} \cdot (-1)^{\frac{1}{2}\#(\mathcal{I}\Delta\mathcal{J})[\#(\mathcal{I}\Delta\mathcal{J})+1]}.\end{aligned}$$

[48]We use the notation $\mathcal{I}^{\complement}$ for the *set complement* of $\mathcal{I}$ with respect to the universal set $\mathrm{bset}(\boldsymbol{\Omega}) = \{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$ or $\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_{n+1}\}$.

Replacing the symmetric difference operator according to the identity $\#(\mathcal{I}\,\Delta\,\mathcal{J}) = \#(\mathcal{I}) + \#(\mathcal{J}) - 2\,\#(\mathcal{I}\cap\mathcal{J})$ leads to

$$\begin{aligned}\sigma(\mathbf{e}_I,\mathbf{e}_J)\cdot\sigma(\mathbf{e}_I\circ\mathbf{e}_J,\mathbf{e}_K) &= (-1)^{[\#(\mathcal{I})+\#(\mathcal{J})+\#(\mathcal{I})\,\#(\mathcal{J})\,\#(\mathcal{I}\cap\mathcal{J})]}\\ &\quad\cdot(-1)^{\frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})[\#(\mathcal{I}\cap\mathcal{J})+1]}\\ &\quad\cdot(-1)^{\frac{1}{2}[\#(\mathcal{I})+\#(\mathcal{J})-2\,\#(\mathcal{I}\cap\mathcal{J})]\cdot[\#(\mathcal{I})+\#(\mathcal{J})-2\,\#(\mathcal{I}\cap\mathcal{J})+1]}.\end{aligned}$$

Multiplying out exponential terms and simplifying produces

$$\begin{aligned}&\sigma(\mathbf{e}_I,\mathbf{e}_J)\cdot\sigma(\mathbf{e}_I\circ\mathbf{e}_J,\mathbf{e}_K) =\\ &\qquad(-1)^{[\#(\mathcal{I})+\#(\mathcal{J})+\#(\mathcal{I})\,\#(\mathcal{J})+\#(\mathcal{I}\cap\mathcal{J})+\#(\mathcal{I})\,\#(\mathcal{J})\,\#(\mathcal{I}\cap\mathcal{J})]}\\ &\qquad\cdot(-1)^{\frac{1}{2}\,\#(\mathcal{I})[\#(\mathcal{I})+1]}\cdot(-1)^{\frac{1}{2}\,\#(\mathcal{J})[\#(\mathcal{J})+1]}\cdot(-1)^{\frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})[\#(\mathcal{I}\cap\mathcal{J})+1]}.\end{aligned}\tag{6.23}$$

I leave to the reader the easy exercise of completing the proof by showing that Equations (6.22) and (6.23) in these last forms are equivalent. Assuming this done, we have proved that, for the set of basis blades $\mathcal{B}\ell_{\mathscr{B}}$, Axiom VIII.2 is satisfied under the $\circledast$ product of the algebra $\sigma\mathcal{OC}_{p,q}$.

This completes the subproof that is part (1) of the proof of Theorem 6.9: the proof that the sigma orientation congruent product of the basis blades in $\mathcal{B}\ell_{\mathscr{B}}$ satisfies the orientation congruent algebra's Hestenes-Sobczyk axioms.

6.2.2. *Proof that the Hestenes-Sobczyk Axioms for Orientation Congruent Algebra Determine the Formula for Sign Factor Function.* Now we present the subproof that is part (2) of the proof of Theorem 6.9: the proof that the orientation congruent algebra's Hestenes-Sobczyk axioms determine the formula for sign factor function.

We begin by examining the consequences for the sign factor function σ of the specific orientation congruent axioms and theorems that are either materially different from their Clifford algebra counterparts or completely new. Our logic is not circular, since we can derive all the following rules without invoking the expression for σ given by Equation (6.3) in Definition 6.5.

Let $\mathbf{e}_I,\mathbf{e}_J\in\mathcal{B}\ell_{\mathscr{B}}$ be arbitrary basis blades with basis subsets $\mathcal{I}=\mathrm{bset}(\mathbf{e}_I)$ and $\mathcal{J}=\mathrm{bset}(\mathbf{e}_J)$ such that $r=\#(\mathcal{I})$ and $s=\#(\mathcal{J})$. Then

(1) by using Axiom VII.1′, the associativity of the orientation congruent outer product, we obtain the rule: if $\mathcal{I}\cap\mathcal{J}=\varnothing$, then $\sigma(\mathcal{I},\mathcal{J})=1=(-1)^0$;
(2) by using Axiom VII.2′, the equality of the orientation congruent square of a blade and the product of the quadratic forms of its vectors, we obtain the rule:
if $\mathcal{I}=\mathcal{J}=\mathcal{I}\cap\mathcal{J}$, then $\sigma(\mathcal{I},\mathcal{J})=(-1)^{\frac{1}{2}r(r-1)}=(-1)^{\frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})[\#(\mathcal{I}\cap\mathcal{J})-1]}$;
(3) and by using Theorem 5.12, the $\mathscr{A}$-universal commutativity of $\boldsymbol{\omega}_{\mathscr{A}}$, we obtain the rule: if $\mathcal{I}\subseteq\mathcal{J}$ and $s=\#(\mathcal{J})$ is odd, then $\sigma(\mathcal{I},\mathcal{J})=(-1)^{[rs+\frac{1}{2}r(r+1)]}=(-1)^{\frac{1}{2}r(r-1)}=(-1)^{\frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})[\#(\mathcal{I}\cap\mathcal{J})-1]}$.

From Rules (2) and (3) above we see that, in general, the sign factor function σ of Definition 6.5 is written as $(-1)^\epsilon$ with an exponent ϵ that contains at least the term $\epsilon_1=\frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})[\#(\mathcal{I}\cap\mathcal{J})-1]$, a mod 4 function of the of the two basis subsets $\mathcal{I}$ and $\mathcal{J}$. We do not, however, need to find another expression for any term in ϵ that is obtained by using Axiom VIII.2, the generalized $\mathscr{A}$-universal commutativity of right $\boldsymbol{\omega}_{\mathscr{A}}$-complementation. Since Axiom VIII.2 involves only the same kind of subset relation that appears in Rule (3) above, it cannot modify the term $\frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})[\#(\mathcal{I}\cap\mathcal{J})-1]$ occurring in ϵ.

Only one other term, ϵ_2, is possible in the exponent ϵ of the sign factor function. That ϵ_2 term is contributed by Rule (1) above. However, because Rule (1) is conditional on the subset relation $\mathcal{I}\cap\mathcal{J}=\varnothing$, any such term cannot modify the term $\epsilon_1=\frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})[\#(\mathcal{I}\cap\mathcal{J})-1]$ contributed by Rule (2) which is conditional on the subset relation $\mathcal{I}=\mathcal{J}=\mathcal{I}\cap\mathcal{J}$. This is because, for nonempty basis subsets $\mathcal{I}$ and $\mathcal{J}$, the relation $\mathcal{I}\cap\mathcal{J}=\varnothing$ is incompatible with the relation $\mathcal{I}=\mathcal{J}=\mathcal{I}\cap\mathcal{J}$. Therefore, ϵ_2 is a mod 2, rather than a mod 4, function. As such, it can be expressed by a sum of a constant and various terms with integer coefficients that are the products of the cardinalities of basis subsets, where all cardinalities of basis subsets are to the first power, not second or higher powers.

Additionally, the term ϵ_2 may, in general, be a function only of the cardinalities of the sets $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{I}\cap\mathcal{J}$. This is because the cardinalities of all *relevant* basis subsets derived from $\mathcal{I}$ and $\mathcal{J}$ by the elementary set theoretic operations intersection, union, and complementation can be computed from the cardinalities of $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{I}\cap\mathcal{J}$ alone. However, the basis subset $\mathcal{I}^{\complement}\cap\mathcal{J}^{\complement}$ derived from $\mathcal{I}$ and $\mathcal{J}$ by elementary set theoretic operations is not relevant for determining ϵ_2. To see this consider that the basis set itself $\mathscr{B}$ (with cardinality $n=p+q$) does not appear independently of $\mathcal{I}$ and $\mathcal{J}$ in any of Rules (1), (2), and (3) above.

We may further characterize the sign factor function σ as symmetric. From Theorem 5.28 in Section 5, the Clifford and orientation product compatibility of compatible blades, we conclude that the sign factor function σ of Definition 6.5 must be a symmetric function of its arguments: $\sigma(\mathcal{I},\mathcal{J})=\sigma(\mathcal{J},\mathcal{I})$ for the basis subsets $\mathcal{I},\mathcal{J}\subseteq\mathscr{B}$ of any basis blades.

We have now determined the sign factor function to be of the form $\sigma(\mathcal{I},\mathcal{J})=(-1)^{\epsilon}$ where $\epsilon=\epsilon_1+\epsilon_2$ is a symmetric function of the cardinalities of the relevant basis subsets $\#(\mathcal{I})$, $\#(\mathcal{J})$, and $\#(\mathcal{I}\cap\mathcal{J})$. Combining the above results, the most general possible expression for ϵ_2 is

$$\begin{aligned}\epsilon_2 = a_0 + a_1[\#(\mathcal{I})+\#(\mathcal{J})] + a_2\,\#(\mathcal{I}\cap\mathcal{J}) + a_3\,\#(\mathcal{I})\,\#(\mathcal{J})&\\ + a_4\,\#(\mathcal{I}\cap\mathcal{J})\,[\#(\mathcal{I})+\#(\mathcal{J})] + a_5\,\#(\mathcal{I}\cap\mathcal{J})\,\#(\mathcal{I})\,\#(\mathcal{J}),&\end{aligned}$$

where the undetermined coefficients a_i are integers. Finally, the most general possible expression for ϵ is

$$\begin{aligned}\epsilon &= \epsilon_1+\epsilon_2\\ &= \frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})[\#(\mathcal{I}\cap\mathcal{J})-1]+\epsilon_2\\ &= \frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})[\#(\mathcal{I}\cap\mathcal{J})-1]\\ &\quad + a_0 + a_1[\#(\mathcal{I})+\#(\mathcal{J})] + a_2\,\#(\mathcal{I}\cap\mathcal{J}) + a_3\,\#(\mathcal{I})\,\#(\mathcal{J})\\ &\quad + a_4\,\#(\mathcal{I}\cap\mathcal{J})\,[\#(\mathcal{I})+\#(\mathcal{J})] + a_5\,\#(\mathcal{I}\cap\mathcal{J})\,\#(\mathcal{I})\,\#(\mathcal{J}),\end{aligned}$$

where the undetermined coefficients a_i are integers. Next we examine the implications of Rules (1), (2), and (3) for the values of these six coefficients.

First, under the condition $\mathcal{I} = \mathcal{J} = \mathcal{I} \cap \mathcal{J}$ of Rule (2) we have

$$\begin{aligned}\epsilon = &\frac{1}{2}\,\#(\mathcal{I} \cap \mathcal{J})[\#(\mathcal{I} \cap \mathcal{J}) - 1]\\ &+ a_0 + a_1[0] + a_2\,\#(\mathcal{I} \cap \mathcal{J}) + a_3\,\#(\mathcal{I})\,\#(\mathcal{J})\\ &+ a_4\,\#(\mathcal{I} \cap \mathcal{J})[0] + a_5\,\#(\mathcal{I} \cap \mathcal{J})\,\#(\mathcal{I})\,\#(\mathcal{J}),\end{aligned}$$

where the undetermined coefficients a_i are integers. Since, in this case, by Rule (2) $\epsilon = \frac{1}{2}\,\#(\mathcal{I} \cap \mathcal{J})[\#(\mathcal{I} \cap \mathcal{J}) - 1]$, it follows that $a_0 = 0$ and $a_2 + a_3 + a_5 = 0 \mod 2$, while a_1 and a_4 remain free.

Next, under the condition $(\mathcal{I} \cap \mathcal{J}) = \varnothing$ of Rule (1) we have

$$\begin{aligned}\epsilon = &\,a_1[\#(\mathcal{I}) + \#(\mathcal{J})] + a_2[0] + a_3\,\#(\mathcal{I})\,\#(\mathcal{J})\\ &+ a_4[0] + a_5[0],\end{aligned}$$

where the coefficients a_i are integers, $a_0 = 0$, and $a_2 + a_3 + a_5 = 0 \mod 2$. Since, in this case, by Rule (1) $\epsilon = 0$, it follows that $a_0 = a_1 = a_3 = 0$ and $a_2 + a_5 = 0 \mod 2$, while a_4 remains free. Therefore, we have

$$\begin{aligned}\epsilon = &\frac{1}{2}\,\#(\mathcal{I} \cap \mathcal{J})[\#(\mathcal{I} \cap \mathcal{J}) - 1]\\ &+ b\,\#(\mathcal{I} \cap \mathcal{J})[\#(\mathcal{I})\,\#(\mathcal{J}) + 1] + a_4\,\#(\mathcal{I} \cap \mathcal{J})[\#(\mathcal{I}) + \#(\mathcal{J})],\end{aligned}$$

where the coefficients $b := a_2 = a_5$ and a_4 are integers.

Finally, under the conditions, $\mathcal{I} \subseteq \mathcal{J}$ (or equivalently $\mathcal{I} \cap \mathcal{J} = \mathcal{I}$) and $s = \#(\mathcal{J})$ is odd, of Rule (3), and after mod 2 simplification, we have

$$\epsilon = \frac{1}{2}\,\#(\mathcal{I})[\#(\mathcal{I}) - 1] + b[0] + a_4\,\#(\mathcal{I}),$$

where the coefficients b and a_4 are integers. Since, in this case, by Rule (3) $\epsilon = \frac{1}{2}\,\#(\mathcal{I})[\#(\mathcal{I}) - 1]$, it follows that $a_4 = 0$, while b remains free.

We have now obtained the following expression for ϵ:

$$\epsilon = \frac{1}{2}\,\#(\mathcal{I} \cap \mathcal{J})[\#(\mathcal{I} \cap \mathcal{J}) - 1] + b\,\#(\mathcal{I} \cap \mathcal{J})[\#(\mathcal{I})\,\#(\mathcal{J}) + 1], \tag{6.24}$$

where only the integral coefficient $b := a_2 = a_5$ remains undetermined. To determine the value of b consider the results $\mathbf{e}_{12} \circ \mathbf{e}_{23} = -\mathbf{e}_{31}$ from Table 5.5 and $\mathbf{e}_{12} \circledcirc \mathbf{e}_{23} = \mathbf{e}_{31}$ from Table 5.6. In this case from Equation (6.24) above we have

$$\epsilon = b[5].$$

On the other hand, from Tables 5.5 and 5.6 we know that $\epsilon = 1 \mod 2$. Therefore, $b = 1 \mod 2$. Thus from the modified Hestenes-Sobczyk axioms for the orientation congruent algebra of a quadratic form given in 5 we have derived the following final equation for ϵ such that the sign factor function of Definition 6.5 is $\sigma = (-1)^{\epsilon}$:

$$\epsilon = \frac{1}{2}\,\#(\mathcal{I} \cap \mathcal{J})[\#(\mathcal{I} \cap \mathcal{J}) - 1] + \#(\mathcal{I} \cap \mathcal{J})[\#(\mathcal{I})\,\#(\mathcal{J}) + 1], \tag{6.25}$$

To finish the second subproof we easily manipulate the expression for ϵ in Equation (6.25) above into the exact form given by Equation (6.3) of Definition 6.5:

$$\begin{aligned}\epsilon &= \#(\mathcal{I}\cap\mathcal{J})\,[\#(\mathcal{I})\,\#(\mathcal{J})+1]+\frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})\,[\#(\mathcal{I}\cap\mathcal{J})-1]\\ &= \#(\mathcal{I}\cap\mathcal{J})\,\#(\mathcal{I})\,\#(\mathcal{J})+\frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})\,[\#(\mathcal{I}\cap\mathcal{J})+1]\\ &= \frac{1}{2}\,\#(\mathcal{I}\cap\mathcal{J})\,[2\,\#(\mathcal{I})\,\#(\mathcal{J})+\#(\mathcal{I}\cap\mathcal{J})+1].\end{aligned}$$

After having now finished both its first and second subproofs, we have completed the proof of Theorem 6.9 that the Clifford-like sigma orientation congruent algebra $\sigma\mathcal{OC}_{p,q}$ and its product $\circledast$ defined by the multilinear extension of Equations (6.2) and (6.3) in Definition 6.5 are identical (isomorphic) to the orientation congruent algebra $\mathcal{OC}_{p,q}$ and its product $\circledcirc$ defined by the primed axioms of Section 5. $\square$

Generally, from now on we drop the word *sigma* and simply refer to the orientation congruent algebra and orientation congruent product, and substitute the symbols $\mathcal{OC}$ and $\circledcirc$ for $\sigma\mathcal{OC}$ and $\circledast$, respectively. However, to explicitly indicate that the orientation congruent product is being computed as the product of the sign factor function and the Clifford product we may refer to the orientation congruent product *in sigma form.*

7. Computer Software Implementations of the Orientation Congruent Algebra

> Indeed, I have to admit my own frustration in not being able to do more than a line or two of computations without making a serious mistake. I believe that what is most needed in the area today is an efficient computer software package for carrying out symbolic calculations in geometric [Clifford] algebra.
>
> Garret Sobczyk [169, p. 18]

Here we give some algorithms for computing the orientation congruent product using existing computer software packages. The practical necessity of computer aided computation of Clifford algebra operations has been noted in the above quote.

By converting the Clifford product to the orientation congruent product, the sign factor function provides a way to compute the later either automatically or manually. The algorithms exploiting this fact that we give here are as simple as possible within the limitations of the software packages used. Except for a few elementary remarks we will not investigate the efficiency of these methods.

Of the many possible computer software packages available we will discuss algorithms for just two prototypical examples: Mathematica and Clical.[49] Mathematica is adaptable to do Clifford algebra calculations through programming; on the other hand, Lounesto's MS-DOS program Clical is specifically designed to do them with built-in functions.

Of the four implementations discussed, only one, using Mathematica, does full-blown, basis-free symbolic manipulation of Clifford or orientation congruent algebra expressions. Although, of course, the other two Mathematica implementations (after major revision) could, but the Clical one could never do so. Nevertheless, all these implementations are useful within their limitations—even those in which multivectors must be expressed as linear combinations of basis blades.

An algorithm suited to Mathematica, which is a completely programmable, symbolic computer algebra system (CAS), will be different than one suited to Clical, which is a numerical software package that can only run scripts without loops or conditional branches. Also Clical is limited to dimensions $n \leq 10$. Consequently, in Mathematica, computation of the orientation congruent product may be done by straightforward translation of the fundamental decomposition in Theorem 7.2 below into a program of nested loops. While in Clical, the loops representing the fundamental decomposition must be *rolled out* into a sum of functions whose number and definition varies with the dimension of the base vector space V^n.

First, we derive the fundamental decomposition theorem of the orientation congruent product in sigma form; using it gives a basic efficiency improvement over an algorithm based on Theorem 6.7. Next, we present two Mathematica implementations that use the fundamental decomposition theorem as well as one that is fully symbolic and basis-optional. Last, we discuss the Clical implementation of the orientation congruent product as a sum of predefined functions.

[49] This software is available online from the sources in Reference [120].

7.1. **The Fundamental Decomposition Theorem of the Orientation Congruent Product.** First we repeat Theorem 6.7 and Corollary 6.8 for easy reference.[50] Then we give the fundamental decomposition theorem for the Clifford product and derive the corresponding theorem for the orientation congruent product from it.

Theorem 6.7. *For any* homogeneous multivectors $A_r, B_s \in \sigma\mathcal{OC}_{p,q}$ *and* $\mathcal{C}\ell_{p,q}$*, with subscripts indicating their grades, the* multilinear extension *of the product* $\circledast$ *of the sigma orientation congruent algebra* $\sigma\mathcal{OC}_{p,q}$ *given by Definition 6.5 is*

$$A_r \circledast B_s = \sum_{t=|r-s|}^{r+s} \langle A_r \circledast B_s \rangle_t = \sum_{t=|r-s|}^{r+s} \sigma_t(r,s)\, \langle A_r \circ B_s \rangle_t, \tag{6.7}$$

where the sign factor function[44] $\sigma_t \colon \mathbb{Z}[0,n] \times \mathbb{Z}[0,n] \mapsto \pm\{\,1\,\}$*, now a function of the grades of* A_r, B_s *and parametrized by the grade* $t \in \mathbb{Z}[0,n]$ *of the* t*-vector part of* $A_r \circ B_s$*, is given by*

$$\sigma_t(r,s) = (-1)^{\frac{1}{8}[r+s-t][4rs+r+s-t+2]}. \tag{6.8}$$

Corollary 6.8. *For all* $A, B \in \sigma\mathcal{OC}_{p,q}$

$$A \circledast B = \sum_{r,s} \langle A \rangle_r \circledast \langle B \rangle_s \text{ as evaluated by Equation (6.7).} \tag{6.9}$$

The Fundamental Clifford Product Decomposition Theorem

Theorem 7.1. *For all homogeneous multivectors* $A_r, B_s \in \mathcal{C}\ell_{p,q}$ *with subscripts indicating their grades their Clifford product may be written as a sum of homogeneous multivectors*

$$\begin{aligned} A_r \circ B_s &= \langle A_r \circ B_s \rangle_{|r-s|} + \langle A_r \circ B_s \rangle_{|r-s|+2} + \cdots + \langle A_r \circ B_s \rangle_{r+s} \\ &= \sum_{k=0}^{m} \langle A_r \circ B_s \rangle_{|r-s|+2k}, \end{aligned} \tag{7.1}$$

where $m = \frac{1}{2}(\mathcal{D}_n(r+s) - |r-s|)$ *with index function*

$$\mathcal{D}_n(i) := \begin{cases} i, & \text{if } 0 \le i \le n, \quad \text{and} \\ 2n - i, & \text{if } n \le i \le 2n. \end{cases} \tag{7.2}$$

Proof. A proof of the infinite n version of Theorem 7.1 is sketched by Hestenes and Sobczyk in their book [97, p. 10]. Harke also mentions it [84, p. 10, Eq. (48)]. This finite n form of the fundamental Clifford product decomposition is found in Conradt's paper [48, Eqs. (3), (4)] or his book contribution [49, Eqs. (3), (4)] □

In the next theorem we display the result of inserting the above formula (6.8) for $\sigma_t(r,s)$ as a multiplier of the grade selected Clifford products in the fundamental decomposition of the Clifford product from Equation (7.1). Theorem 7.2 presents the *fundamental decomposition of the orientation congruent product in terms of the sign factor function* $\sigma_t(r,s)$ and the Clifford product (or, briefly, in *sigma form*).

[50]In reading this theorem and corollary please recall that after proving the algebra isomorphism Theorem 6.9 we have now dropped the word *sigma* to leave simply *orientation congruent* and substituted the symbols $\mathcal{OC}$ and $\circledcirc$ for $\sigma\mathcal{OC}$ and $\circledast$, respectively.

The Fundamental $\mathcal{OC}$ Product Decomposition Theorem in sigma form

Theorem 7.2. *For all homogeneous multivectors $A_r, B_s \in \mathcal{OC}_{p,q}$ and $\mathcal{C}\ell_{p,q}$, with subscripts indicating their grades, their orientation congruent product may be written as a sum of homogeneous multivectors*

$$\begin{aligned} A_r \circledcirc B_s = \ & \sigma_{|r-s|}(r,s)\, \langle A_r \circ B_s \rangle_{|r-s|} \\ & + \sigma_{|r-s|+2}(r,s)\, \langle A_r \circ B_s \rangle_{|r-s|+2} + \cdots \\ & + \sigma_{r+s}(r,s)\, \langle A_r \circ B_s \rangle_{r+s} \\ = & \sum_{k=0}^{m} \sigma_{|r-s|+2k}(r,s)\, \langle A_r \circ B_s \rangle_{|r-s|+2k} \end{aligned} \tag{7.3}$$

where the summation limit m and the index function $\mathcal{D}_n(i)$ are the same as for Equation (7.1)*, and $\sigma_t(r,s)$ is given by Equation* (6.8) *in Theorem 6.7.*

Proof. The proof is immediate from Lemma 6.6 and Theorem 7.1 by the multilinearity of the Clifford product and the linearity of the grade selection operator. □

The number of grade selections (and consequent orientation congruent product evaluations) is reduced by at least $\min(r,s)$ when Equation (7.3) from Theorem 7.2 above is employed instead of Equation (6.7). Using the index function defined in the theorem, $\mathcal{D}_n(i)$, to determine the upper summation limit, m, reduces the number of products computed even more than the lower bound of $\min(r,s)$.

The Mathematica function `OCpD` given below in Figure 7.1 achieves this maximum efficiency. `OCpD` is defined in terms of functions from the package Clifford which does *not* require a dimension n to be declared. Therefore, the parameter n of $\mathcal{D}_n(i)$ is set equal to the highest index of any of basis vectors in A_r or B_s. Using this value for n has exactly the same effect on the computational efficiency of a fundamental decomposition based algorithm as would using a value that is the dimension of any base space V^n that allows both A_r and B_s to be nonzero.

However, this is possible only when multivectors are expressed as linear combinations of basis blades, as is done in the package Clifford. If the dimension n is not fixed or known, using a basis-free algorithm based on the fundamental decomposition extracts a penalty of inefficiency. Then we must fall back on the least efficient basis-free strategy, abandoning the index function and simply setting $m = r + s$. Still, in comparison with Equation (6.7), the number of evaluations of Clifford products is reduced by $\min(r,s)$ in absolute terms. In the limit of infinite $\min(r,s)$, the fractional reduction is one half.

7.2. **$\mathcal{OC}$ Computations in Mathematica with *Clifford*.** I have programmed Equation (7.3) in a Mathematica notebook as the external function `OCpD` of Figure 7.1. This function and some auxiliary ones (not given) are based on a slightly modified version of the existing package *Clifford*. This package is internally titled "Clifford Algebra of a Euclidean Space" by its authors Oscar G. Caballero and José Luis Aragón Vera.[51] It computes Clifford algebra and quaternion operations in terms of the basis blades constructed from an orthonormal set of basis vectors denoted by `e[1]`, `e[2]`, ..., `e[n]`.

[51] This package is available online in two versions from the sources in References [38] and [39].

```
(* ************************************************* *)
(*                 Define OCpD ver. 1                *)
(*   Orien. Cong. Product in Fund. Decomposition Form   *)
(* ************************************************* *)
ClearAll[OCpD]
Remove[OCpD]
OCpD[x_, y_] := Module[{xGradeMin, xGradeMax, yGradeMin, yGradeMax,
    xyDimMax, Dind, TempSum, r, s, k},
  xGradeMin = GradeMin[x];  xGradeMax = GradeMax[x];
  yGradeMin = GradeMin[y];  yGradeMax = GradeMax[y];
  xyDimMax = Max[DimMax[x], DimMax[y]];
  Dind[i_Integer, n_Integer] :=
  Which[
    0 <= i && i <    n, i,
    n <= i && i <= 2 n, 2 n - i
  ];
  TempSum = 0;
  r = xGradeMin;
  While[r <= xGradeMax,
    s = yGradeMin;
    While[s <= yGradeMax,
      k = Abs[r-s];
      While[k <= Dind[r+s, xyDimMax],
        TempSum += SFac[r,s,k] Grade[Gp[Grade[x,r], Grade[y,s]],k];
        k = k + 2
      ];
      s = s + 1
    ];
    r = r + 1
  ];
  TempSum
]
```

Figure 7.1. External Mathematica Function `OCpD`. This function gives the $\mathcal{OC}$ product based on the fundamental decomposition theorem, Theorem 7.2.

I have also programmed Equation (7.3) as a Mathematica function internally defined within an altered version of the Caballero and Aragón Vera package *Clifford*. This function `OCp` (Figure 7.2 below) is a directly modified form of the package's definition of `Gp`. It computes the orientation congruent product by a straightforward use of the sign factor function as a multiplier defined by the assignment `sff=(-1)^(gu (2 g1 g2 + gu + 1)/2)`. Since the loops needed to implement Theorem 7.2 are already built into the definition of `Gp`, the `OCp` function runs much more quickly than the external function `OCpD` of Figure 7.1.

```
(* Begin OC Product Section *)
OCProduct[ _] := $Failed
OCProduct[m1_,m2_,m3__] := tmp[OCProduct[m1,m2],m3] /.
            tmp->OCProduct
OCProduct[m1_,m2_] := ocprod[Expand[m1],Expand[m2]] //
                     Expand
(* The next 3 assignments define the alias OCp. *)
OCp[ _] := $Failed
OCp[m1_,m2_,m3__] := tmp[OCp[m1,m2],m3] /.
            tmp->OCp
OCp[m1_,m2_] := ocprod[Expand[m1],Expand[m2]] //
                     Expand
ocprod[a_,y_] := a y /; FreeQ[a,e[_?Positive]]
ocprod[x_,a_] := a x /; FreeQ[a,e[_?Positive]]
ocprod[x_,y_] := Module[{
  p1=ntuple[x,Max[dimensions[x],dimensions[y]]],q=1,s,r={},r1={},
 p2=ntuple[y,Max[dimensions[x],dimensions[y]]],
 g1=grados[x],g2=grados[y],gu,sff},
 gu=p1.p2;
 sff=(-1)^(gu (2 g1 g2 + gu + 1)/2);
 s=Sum[p2[[m]]*p1[[n]],{m,Length[p1]-1},{n,m+1,Length[p2]}];
     r1=p1+p2;
       r=Mod[r1,2];
            Do[ If[r[[i]] == 1, q *= e[i]];
     If[r1[[i]] == 2,
       q *= bilinearform[e[i],e[i]]],{i,Length[r1]} ];
    (-1)^s*q*sff ]
ocprod[a_ x_,y_] := a ocprod[x,y] /; FreeQ[a,e[_?Positive]]
ocprod[x_,a_ y_] := a ocprod[x,y] /; FreeQ[a,e[_?Positive]]
ocprod[x_,y_Plus] := Distribute[tmp[x,y],Plus] /. tmp->ocprod
ocprod[x_Plus,y_] := Distribute[tmp[x,y],Plus] /. tmp->ocprod
(* End of OC Product Section *)
```

FIGURE 7.2. **Internal Mathematica Function** `OCp`**.** This defines the $\mathcal{OC}$ product as a modified version of the *Clifford* package's definition of the function `Gp`.

7.3. **$\mathcal{OC}$ Computations in Mathematica with *GrassmannAlgebra*.** John Browne has developed the Mathematica package *GrassmannAlgebra* [30] to translate the many operations of Hermann Grassmann's *calculus of extension* into a modern computer system. This powerful package provides a fully symbolic CAS that allows, but does not require, the use of a basis and that can accept general metrics.

John Browne has derived the following function for the orientation congruent product [32]. It is based on the *generalized Grassmann product* $\underset{\lambda}{\Delta}$ of the package.

$$\underset{m}{\alpha} \underset{\lambda}{\circledcirc} \underset{k}{\beta} = \sum_{\lambda=0}^{\mathrm{Min}[m,k]} (-1)^{m\lambda(k+1)} \left(\underset{m}{\alpha} \underset{\lambda}{\Delta} \underset{k}{\beta} \right) \tag{7.4}$$

In Browne's package and book [31] the $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ above are called *elements* (of a multilinear space). This term refers to a general multilinear object, but it implies that the object is not specifically given a geometric interpretation.

Using a general metric, Browne has also demonstrated the facility of his package for transforming the entries in the multiplication table of $\mathcal{OC}_3$ into expressions containing the exterior product and the various forms of inner product available in *GrassmannAlgebra* [32]. His presentation of these results in a Mathematica notebook required 35 pages to print onto letter size paper.

7.4. **$\mathcal{OC}$ Computations in Clical.** The orientation congruent product may also be calculated in Clical, although much less elegantly than in Mathematica, by rolling out the nested loops of a program based on its fundamental decomposition. Let $A_r, B_s \in \mathcal{OC}_{p,q}$ be blades. Then the fundamental decomposition theorem, Theorem 7.2, states that the product $A_r \circledcirc B_s$ is not necessarily homogeneous. This theorem is naturally parametrized by the pair of grades (r, s) of A_r and B_s.

However, the tables below, and the functions derived from them, are instead naturally parametrized by the dimension $n = p + q$ of the base vector space of a given Clifford algebra. This is because in Clical the dimension of the Clifford algebra $\mathcal{C}\ell_{p,q}$ in which one will calculate is fixed by first declaring its signature (p, q). Also the sign factor function σ is dependent on three grades: r, s, and t, where t is the grade of the t-vector part of the product, $\langle A_r \circledcirc B_s\rangle_t$.

Therefore, for Clical we define a sequence of *winnow* functions each of which is a sum of terms of the form $\sigma_t(r, s)\,\langle A_r \circ B_s\rangle_t = \langle A_r \circledcirc B_s\rangle_t$. The *order* of one of these functions is defined to be the lowest dimension that allows all its terms to be (potentially) nonzero. Then, the sum of these functions up to order n contains only the grade-selected parts of the product $\langle A_r \circledcirc B_s\rangle_t$ that are *just* permitted by the dimension n. Therefore, in general, the summands $\sigma_{|r-s|+2k}(r, s)\,\langle A_r \circ B_s\rangle_{|r-s|+2k}$ in the fundamental decomposition of $A_r \circledcirc B_s$ in sigma form for a given r and s appear in *several* winnow functions of *different* orders.

Let the parameter t represent the selected grade of an orientation congruent (or Clifford) product as in $\langle A_r \circledcirc B_s\rangle_t$. Then, quite generally,[52] we define the integer k to be one half the reduction of the grade of the product from the sum of the grades of its factors:

$$k := \frac{1}{2}(r + s - t). \tag{7.5}$$

We may also express this relationship as

$$r + s = t + 2k. \tag{7.6}$$

[52]Equation (7.5) is seen to be the natural generalization of Equation (6.5b) with Equation (6.6) applied to it, if we let $A_r = \mathbf{e}_i$ and $B_s = \mathbf{e}_j$ with $\mathbf{e}_i, \mathbf{e}_j \in \mathscr{B}^\wedge$, and $\mathcal{A} = \mathrm{bset}(\mathbf{e}_i)$, $\mathcal{B} = \mathrm{bset}(\mathbf{e}_j)$, and $\mathcal{C} = \mathrm{bset}(\mathbf{e}_i \circledcirc \mathbf{e}_j)$, and $k = \#\,\mathcal{A} \cap \mathcal{B}$, $r = \#\,\mathcal{A}$, $s = \#\,\mathcal{B}$, and $t = \#\,\mathcal{C}$.

As a guide to defining the sequence of winnow functions introduced above we construct tables, one for each integer $m \geq 0$, that display the values of t, k, $r+s$, and the pairs of grades of factors (r, s), *whose products* $\langle A_r \circ B_s \rangle_t$ *may first become nonzero when the dimension of the base vector space* n *is equal to* m. The rows in these tables are ordered from top to bottom by increasing t. We order the pairs of grades (r, s) in a row from left to right by increasing r. Of course, we also require that all values in these tables satisfy $t, k, r, s \in \mathbb{Z}[0, m]$.[53]

Four examples of these tables are given below as Tables 7.1, 7.2, 7.3, and 7.4 for m equal to 2, 3, 4, and 5, respectively. We will ignore the lining out of some terms; this, as well as the use of bold fonts, will be explained later.

Table 7.1. The grades of factors and products that first may be nonzero when the dimension $n = m = 2$. The text explains the lined out pairs of factor grades.

t	k	$r+s$	(r,s)		
0	2	4	$\cancel{(2,2)}$		
1	1	3	$\cancel{(1,2)}$	$\cancel{(2,1)}$	
2	0	2	$\cancel{(0,2)}$	$\cancel{(1,1)}$	$\cancel{(2,0)}$

Table 7.2. The grades of factors and products that first may be nonzero when the dimension $n = m = 3$. The text explains the lined out pairs of factor grades.

t	k	$r+s$	(r,s)			
0	3	6	$\cancel{(3,3)}$			
1	2	5	$\cancel{(2,3)}$	$\cancel{(3,2)}$		
2	1	4	$\cancel{(1,3)}$	$\cancel{\mathbf{(2,2)}}$	$\cancel{(3,1)}$	
3	0	3	$\cancel{(0,3)}$	$\cancel{(1,2)}$	$\cancel{(2,1)}$	$\cancel{(3,0)}$

Table 7.3. The grades of factors and products that first may be nonzero when the dimension $n = m = 4$. The text explains the lined out pairs of factor grades.

t	k	$r+s$	(r,s)				
0	4	8	$\cancel{(4,4)}$				
1	3	7	$\cancel{(3,4)}$	$\cancel{(4,3)}$			
2	2	6	$\cancel{(2,4)}$	$(3,3)$	$\cancel{(4,2)}$		
3	1	5	$\cancel{(1,4)}$	$\cancel{\mathbf{(2,3)}}$	$\cancel{\mathbf{(3,2)}}$	$\cancel{(4,1)}$	
4	0	4	$\cancel{(0,4)}$	$\cancel{(1,3)}$	$\cancel{(2,2)}$	$\cancel{(3,1)}$	$\cancel{(4,0)}$

The italicized clause occurring two paragraphs up may be put another way: for one of these tables m is the *minimum value* of the dimension n that permits any row to exist; that is, that allows all grade selected products $\langle A_r \circ B_s \rangle_t$ resulting

[53]Here we have used the convenient notation $\mathbb{Z}[a, b] := \{\, i \mid i \in \mathbb{Z} \text{ and } a \leq i \leq b \,\}$.

TABLE 7.4. The grades of factors and products that first may be nonzero when the dimension $n = m = 5$. The text explains the lined out pairs of factor grades.

t	k	$r+s$	(r,s)					
0	5	10	$\cancel{(5,5)}$					
1	4	9	$\cancel{(4,5)}$	$\cancel{(5,4)}$				
2	3	8	$\cancel{(3,5)}$	$(4,4)$	$\cancel{(5,3)}$			
3	2	7	$\cancel{(2,5)}$	$(3,4)$	$(4,3)$	$\cancel{(5,2)}$		
4	1	6	$\cancel{(1,5)}$	$\cancel{\mathbf{(2,4)}}$	$(3,3)$	$\cancel{\mathbf{(4,2)}}$	$\cancel{(5,1)}$	
5	0	5	$\cancel{(0,5)}$	$\cancel{(1,4)}$	$\cancel{(2,3)}$	$\cancel{(3,2)}$	$\cancel{(4,1)}$	$\cancel{(5,0)}$

from homogeneous factors with grades r and s given by all pairs displayed in a row to be, in general, nonzero. Then it is easily seen that for each row in the m-table

$$m = r + s - k. \tag{7.7a}$$

Applying Equation (7.6) we obtain

$$m = t + k.^{54} \tag{7.7b}$$

These two equations may be rearranged to also give

$$r + s = m + k \quad \text{and} \tag{7.8a}$$

$$t = m - k. \tag{7.8b}$$

Adding the last two equations and rearranging yields

$$t = 2m - (r + s). \tag{7.9}$$

We recognize the last equation as $t = \mathcal{D}_m(r + s)$ after applying the second line of the index function $\mathcal{D}_n(i)$ definition in Equation (7.2) from the fundamental decomposition theorem. This leads directly to the observation that each table is constructed so that $m \leq t \leq 2m$.

This is also why, in general, a given pair of factor grades tracks along a course of consecutive tables. Specifically, in agreement with the fundamental decomposition theorem, if the pair (r, s) occurs in position (i, j) in the m-table, it also appears in position $(i + 2, j + 1)$ in the $(m + 1)$-table, if $m + 1 \leq r + s \leq 2(m + 1)$. (Here we have anticipated the "matrix" interpretation of the next paragraph.)

The pairs of factor grades in each table may be indexed as $(r, s)_{i,j}$ so that they constitute a "matrix"[55] of ordered pairs in the last column of that table. Each row of $[(r, s)_{i,j}]$ is aligned with the corresponding values of the parameters t, k, and $r + s$.

The row and column indices of this matrix satisfy $i, j \in \mathbb{Z}[1, m + 1]$. The row index may be written in terms of the row parameter k by

$$i = m - k + 1. \tag{7.10}$$

[54]Equations (7.7a) and (7.7b) are the Clifford algebra analogues of the set-theoretic formulas $\#\mathcal{A} \cup \mathcal{B} = \#\mathcal{A} + \#\mathcal{B} - \#\mathcal{A} \cap \mathcal{B}$, and $\#\mathcal{A} \cup \mathcal{B} = \#\mathcal{A} \,\Delta\, \mathcal{B} + \#\mathcal{A} \cap \mathcal{B}$, respectively.

[55]Properly, of course, the objects containing these indexed pairs should be called *indexed tables* or *arrays,* since we are not defining matrix addition (let alone multiplication) for them. Also, we let missing entries in the table become doubly 0-valued entries $(0, 0)$ in the "matrix."

Nonzero entries of each row of the matrix $[(r,s)_{i,j}]$ must satisfy $\min(r,s) \geq k$, in addition to the already derived $0 \leq r, s \leq m$ and $r+s = m+k$ with $r+s \in \mathbb{Z}[m, 2m]$. All matrix entries, including invalid ones that should be zero, are given in terms of the row parameter k and the column index j by

$$(r,s)_{i,j} = (k+j-1, m-j+1)_{i,j}. \tag{7.11}$$

Solving Equation (7.10) for k and substituting in the first half of the pair on the right hand side of Equation (7.11) gives

$$(r,s)_{i,j} = (m-i+j, m-j+1)_{i,j}. \tag{7.12}$$

Requiring that the first half of the pair on the right hand side of the last equation satisfies $0 \leq r, s \leq m$ yields

$$j \leq i, \tag{7.13}$$

which expresses that the matrix $[(r,s)_{i,j}]$ is naturally *lower triangular.*

We now begin to define, as an example, a sequence of winnow functions whose sum is the orientation congruent product in an algebra of base dimension $m = n = p+q = 5$. These definitions are valid for all multivector arguments $A, B \in \mathcal{OC}_{p,q}$. We denote this product in the functional form $\texttt{oc}(A,B)$ similar to the way it would appear in Clical. Clical provides the grade selection operator which we need. But, we write it in the usual way with angular brackets and a subscript indicating the grade r to be selected as $\langle A \rangle_r$ rather than as it would be written in Clical as $\texttt{Pu}(r, A)$.

It is convenient to start by defining two *base* winnow functions that include terms that first become nonzero at a variety of dimensions. As such they are of *inhomogeneous* order and may be called simply *base functions.* The first of these base functions, $\texttt{ocbaseone}(A,B)$, contains terms of lowest order zero; while the second, $\texttt{ocbasetwo}(A,B)$, contains terms of lowest order three. The functions of homogeneous order start with $\texttt{ocdimfour}(A,B)$ which as its name suggests is of order four.

For the definition of $\texttt{ocbaseone}(A,B)$ we need the orientation congruent left and right *contraction operators,* $\neg$ and $\ulcorner$, respectively. These may be defined by the following equations[56] written in terms of some operations and a constant[57] that are all available in Clical. Here both $\boldsymbol{I}$ and $\texttt{j}$ represent the Clifford algebra pseudoscalar.

$$\begin{aligned} A \neg B &= \boldsymbol{I}^{-1} \circ [(\boldsymbol{I} \circ B) \wedge A^{\dagger}] && \text{(in normal notation)} \\ \texttt{oclcont}(\texttt{A},\texttt{B}) &= \texttt{j} \setminus ((\texttt{j} * \texttt{B}) \,\hat{}\, \texttt{A}\tilde{}\,) && \text{(as in Clical)} \end{aligned} \tag{7.14}$$

$$\begin{aligned} A \ulcorner B &= [B^{\dagger} \wedge (A \circ \boldsymbol{I})] \circ \boldsymbol{I}^{-1} && \text{(in normal notation)} \\ \texttt{ocrcont}(\texttt{A},\texttt{B}) &= (\texttt{B}\tilde{}\,\hat{}\,(\texttt{A} * \texttt{j}))/\texttt{j} && \text{(as in Clical)} \end{aligned} \tag{7.15}$$

The first winnow function $\texttt{ocbaseone}(A,B)$ contains the (possibly null) terms that, for $A_r \circledcirc B_s$ with homogeneous operands, *are of extremum grade* $|r-s|$ *or* $r+s$

[56]These Equations (7.14) and (7.15) are derived and proved valid in Section 8. See Table 8.6, line (8).

[57]The constant `j` in these function definitions is predefined in Clical only for algebras $\mathcal{C}\ell_{p,q}$ of dimension $n = p+q \geq 3$. Clical predefines another constant `i` for $n \leq 2$. The following Clical script defines a variable `jj` which is the pseudoscalar in any dimension Clical can handle, $0 \leq n \leq 10$ (semicolons are used here to indicate the end of a Clical script line): `jj = 0; jj = j; jj = jj + i;`

in the fundamental decomposition of the orientation congruent product. In other words, it contains all orientation congruent inner and outer product terms found in the orientation congruent product of general multivectors for any dimension n. In particular, the orientation congruent products for dimensions $n \leq 2$ are completely contained in it.

The base winnow function $\texttt{ocbaseone}(A, B)$ is defined by[58]

$$\begin{aligned}\texttt{ocbaseone}(A,B) := {} & + A \neg B + A \ulcorner B - \frac{1}{2}\langle A \neg B + A \ulcorner B\rangle_0 \\ & + (A - \langle A\rangle_0) \wedge (B - \langle B\rangle_0).\end{aligned} \tag{7.16}$$

The Clifford product *commutator* $\texttt{clcom}(A, B)$ is used in the definition the next winnow function. The definition of the commutator is valid for any dimension n and is given by

$$\texttt{clcom}(A,B) := \frac{1}{2}(A \circ B - B \circ A). \tag{7.17}$$

The second winnow function $\texttt{ocbasetwo}(A, B)$ includes all terms of the orientation congruent product decomposition *that are not contained in the base winnow function* $\texttt{ocbaseone}(A, B)$ and that result from a product of factors at least one of which is of grade two. Accordingly, it may be nonzero only when $n \geq 3$. The commutator excludes all terms of orientation congruent products that are also orientation congruent inner or outer products; these are already included in $\texttt{ocbaseone}(A, B)$. The commutator neatly replaces grade selection for this purpose.

The base winnow function $\texttt{ocbasetwo}(A, B)$ is defined by

$$\begin{aligned}\texttt{ocbasetwo}(A,B) := {} & - \texttt{clcom}(A, \langle B\rangle_2) - \texttt{clcom}(\langle A\rangle_2, B) \\ & + \texttt{clcom}(\langle A\rangle_1, \langle B\rangle_2) + \texttt{clcom}(\langle A\rangle_2, \langle B\rangle_1) \\ & + \texttt{clcom}(\langle A\rangle_2, \langle B\rangle_2),\end{aligned} \tag{7.18a}$$

or, equivalently,

$$\begin{aligned}\texttt{ocbasetwo}(A,B) := {} & - \texttt{clcom}(A - \langle A\rangle_1 - \frac{1}{2}\langle A\rangle_2, \langle B\rangle_2) \\ & - \texttt{clcom}(\langle A\rangle_2, B - \langle B\rangle_1 - \frac{1}{2}\langle B\rangle_2).\end{aligned} \tag{7.18b}$$

We digress to explain the lined out pairs in $[(r, s)_{i,j}]$. These are simply the pairs of factors whose grade-selected product is either an inner product (in the first column or the main diagonal) or an outer product (in the last row),[59] together with those pairs of factors at least one of which is of grade two. In other words, these are all pairs of factors whose terms are included in the base winnow functions $\texttt{ocbaseone}(A, B)$ or $\texttt{ocbasetwo}(A, B)$. In addition to lining out, a bold font is used for the pairs of factors whose products are in $\texttt{ocbasetwo}(A, B)$. The terms resulting from these lined out factor pairs must be excluded from the higher order functions we define next.

The statements above about which matrix entries are lined out may also be expressed algebraically in terms of r and s in the following complementary form.

[58] Other equivalent expressions may serve as the definition of the function $\texttt{ocbaseone}(A, B)$.

[59] The $(r, s)_{m,1}$ and $(r, s)_{m,m}$ entries with a scalar part are pairs of factors whose orientation congruent product is at the same time both an orientation congruent inner and outer product.

The factor pairs in $[(r,s)_{i,j}]$ that are *not* lined out must satisfy the additional conditions

$$r, s > 2, \tag{7.19a}$$

$$r, s < m, \text{ and} \tag{7.19b}$$

$$r + s > m. \tag{7.19c}$$

The winnow function of order 4 $\texttt{ocdimfour}(A,B)$ holds all terms of the orientation congruent product decomposition *that are not contained in the base winnow functions* and that first may be nonzero when $n = 4$. It is defined by

$$\texttt{ocdimfour}(A,B) := -\langle\langle A\rangle_3 \circ \langle B\rangle_3\rangle_2. \tag{7.20}$$

As an example calculation we find the sign in Equation (7.20) by evaluating the sign factor function in Equation (6.8), repeated here,

$$\sigma_t(r,s) = (-1)^{\frac{1}{8}[r+s-t][4rs+r+s-t+2]} \tag{6.8$'$}$$

with the values in Equation (7.20) above. This gives

$$\begin{aligned} &= (-1)^{\frac{1}{8}[3+3-2][4\cdot 3\cdot 3+3+3-2+2]} \\ &= (-1)^{\frac{1}{8}[4][42]} = (-1)^{21} = -1. \end{aligned}$$

The winnow function of order 5 $\texttt{ocdimfive}(A,B)$ comprises all terms of the orientation congruent product decomposition *that are not contained in the base or lower order winnow functions* and that first may be nonzero when $n = 5$. It is defined by

$$\begin{aligned} \texttt{ocdimfive}(A,B) := &+ \langle\langle A\rangle_3 \circ \langle B\rangle_3\rangle_4 \\ &- \langle\langle A\rangle_4 \circ \langle B\rangle_3\rangle_3 - \langle\langle A\rangle_3 \circ \langle B\rangle_4\rangle_3 \\ &+ \langle\langle A\rangle_4 \circ \langle B\rangle_4\rangle_2. \end{aligned} \tag{7.21}$$

Finally, summing all the above winnow functions (the base functions and the winnow functions of order $m \le 5$) gives $\texttt{oc}(A,B)$ which contains *all* terms of the orientation congruent product decomposition that could be nonzero when $n = 5$ (as well as some that could be nonzero when $n > 5$).

$$\begin{aligned} \texttt{oc}(A,B) := &\ \texttt{ocbaseone}(A,B) + \texttt{ocbasetwo}(A,B) \\ &+ \texttt{ocdimfour}(A,B) + \texttt{ocdimfive}(A,B). \end{aligned} \tag{7.22}$$

We end this section by deriving a formula for T_m the number of terms in a winnow function of order $m \ge 4$. First, consider the number of terms, lined out or not, in a table of order m. Since there are $m+1$ rows in an m-order table and since it has a triangular shape this is just the sum of the first $m+1$ positive integers

$$S_{m+1} = \frac{1}{2}(m+1)(m).$$

If we remove a count of the pairs that give rise to the outer product, those in the last row of the table, and a count of the pairs that give rise to the inner product, those in the first column and on the main diagonal, we get the following formula for the sum of the first $m-1$ positive integers

$$S_{m-1} = \frac{1}{2}(m-1)(m-2).$$

Finally, we remove a count of the pairs with $r = 2$ or $s = 2$. Pairs with at least one 2 in either position always occur in the last three rows of a table of order $m \geq 4$. The pairs in the highest and lowest of these three rows are already excluded because the are in $\texttt{ocbaseone}(A, B)$. In a table of order $m \geq 4$ the middle row always contains exactly two such pairs, neither of which are in $\texttt{ocbaseone}(A, B)$. Thus, we must remove a count of two from S_{m-1}. Therefore, the formula for T_m, the number of terms in a winnow function of order $m \geq 4$, is given by

$$T_m = \frac{1}{2}(m-1)(m-2) - 2. \tag{7.23}$$

8. Inner Products, Contraction Operators, and Duality

Add a subsection on the geometric interpretation of the orientation congruent contraction operators similar to Dorst's [61, p. 43].

8.1. The Significance of the Contraction Operators.

The left, $\rfloor$, and right, $\lfloor$, *contraction operators* of Clifford algebra are of theoretical and practical significance. They are used theoretically, for example, by Fernández et al. [69, p. 15] in an axiomatic exposition of the Clifford algebra $\mathcal{C}\ell_n$. Their approach exploits the following *Cartan decomposition formula* for the Clifford product of a vector and multivector to deform the exterior algebra:

$$\mathbf{x} \circ u = \mathbf{x} \wedge u + \mathbf{x} \rfloor u \quad \text{for all } \mathbf{x} \in V \text{ and all } u \in \bigwedge V. \tag{8.1}$$

In the work of Fernández et al., as well as this paper, the Cartan decomposition formula is explicitly or implicitly the basis for a calculating method for the algebra considered.[60] We say "calculating method" because this approach does not give a genuine axiomatization of $\mathcal{OC}_{p,q}$. Since under a basis change it is does not respect the grading of the elements of the exterior algebra, this method falls short of an axiomatic definition [51, p. 45]. Therefore, our GR axioms for the Clifford and orientation congruent algebras of a nondegenerate quadratic form, strictly speaking, define only a calculating scheme for or representation of these algebras' products in terms of the exterior product and the algebras' contraction operators.

However, this type of axiomatic approach is related to the more fundamental one of Chevalley who embeds the Clifford algebra as a subalgebra of the associated exterior algebra's endomorphism algebra through the Chevalley-operator representation (which Chevalley [44] based on the Cartan decomposition formula). In either approach the contraction operators are crucial.

Lounesto [123, pp. 288–90] discusses the contraction operators while constructing the linear isomorphism $\bigwedge V \to \mathcal{C}\ell(Q)$. On the practical side Lounesto [122, pp. 143 f.] points out the awkwardness of substituting the more symmetrical dot product of Hestenes and Sobczyk [97, p. 6] in constructing proofs. Also Dorst in References [60, p. 10], and [61, p. 47] repeats Lounesto's complaint as well as discusses the difficulties removed by using the contraction operators rather than the Hestenes dot product in designing computer algebra systems for Clifford algebra. For more motivational material see the references cited above as well as Lounesto [121].

Because of the importance of the contraction operators, we present them for both the Clifford and orientation congruent algebras. We give a parallel exposition so that comparison between the two tracks may aid the reader's understanding.

[60] Of course, for the $\mathcal{OC}$ algebra of this paper we would need to substitute the left orientation congruent contraction operator for the Clifford one in Equation (8.1).

8.2. **Fundamental Definitions of the Contraction Operators.** Some introductory text.

Using the Clifford product, various researchers have defined a great number inner products. For a survey of many of these inner products see Ian Bell's website [15]. However, we favor only three Clifford algebra inner products and their orientation congruent analogues. We do define one other Clifford algebra inner product, the Hestenes inner product, but only to translate results in the geometric algebra literature to our preferred inner products. In this paper we do not directly use either the Clifford algebra Hestenes inner product or its orientation congruent algebra analogue.

The *Hestenes inner product* is usually written using a small centered dot. It is designed (particularly for $B \in \mathbb{R}$) to be additively complementary to the outer product by satisfying the equations

$$\mathbf{a} \circ B = \mathbf{a} \wedge B + \mathbf{a} \cdot B \quad \text{and}$$
$$B \circ \mathbf{a} = B \wedge \mathbf{a} + B \cdot \mathbf{a}$$

for all vectors $\mathbf{a} \in V^n$ and all multivectors $B \in \mathcal{C}\ell_{p,q}$ [97, p. 8], [84, pp. 8, 10]. Since the outer product of a scalar and any multivector is equivalent to vector space scalar multiplication, the Hestenes inner product must be restricted to a zero result when either operand is a scalar [84, p. 6, Eq. (18)]. Thus, we may define it formally as follows [97, p. 6], [84, p. 6].

Definition 8.1. For all multivectors $A, B \in \mathcal{C}\ell_{p,q}$

$$A \cdot B := \sum_{r,s>0} \langle \langle A \rangle_r \circ \langle B \rangle_s \rangle_{|s-r|} + 0. \quad \text{Hestenes inner product} \tag{8.2}$$

For both the Clifford algebra and the orientation congruent algebra we prefer three kinds of inner product: the so-called *fat dot inner product* or *modified Hestenes inner product* [16, 61], [122, pp. 143 f.], the *left contraction operator*, and the *right contraction operator*. Unlike the Hestenes inner product, the fat dot inner product is not restricted to zero when either operand is a scalar. Recall that we have adopted the convention that the negative grade projection of any multivector is null: $\langle A \rangle_r := 0$ for all $r < 0$. Then the Clifford algebra versions of these three inner products are defined by the following equations.

Definition 8.2. For all multivectors $A, B \in \mathcal{C}\ell_{p,q}$

$$A \bullet B := \sum_{r,s} \langle \langle A \rangle_r \circ \langle B \rangle_s \rangle_{|s-r|}, \quad \text{Clifford fat dot inner product} \tag{8.3}$$

$$A \,\lrcorner\, B := \sum_{r,s} \langle \langle A \rangle_r \circ \langle B \rangle_s \rangle_{s-r}, \quad \text{left Clifford contraction} \tag{8.4}$$

$$A \,\llcorner\, B := \sum_{r,s} \langle \langle A \rangle_r \circ \langle B \rangle_s \rangle_{r-s}. \quad \text{right Clifford contraction} \tag{8.5}$$

The orientation congruent algebra versions of these three inner products are defined analogously to the Clifford algebra ones by the following equations.

Definition 8.3. For all multivectors $A, B \in \mathcal{OC}_{p,q}$

$$A \odot B := \sum_{r,s} \langle \langle A \rangle_r \circledcirc \langle B \rangle_s \rangle_{|s-r|}, \quad \mathcal{OC} \text{ fat dot inner product} \tag{8.6}$$

$$A \lrcorner B := \sum_{r,s} \langle \langle A \rangle_r \circledcirc \langle B \rangle_s \rangle_{s-r}, \quad \text{left } \mathcal{OC} \text{ contraction} \tag{8.7}$$

$$A \llcorner B := \sum_{r,s} \langle \langle A \rangle_r \circledcirc \langle B \rangle_s \rangle_{r-s}. \quad \text{right } \mathcal{OC} \text{ contraction} \tag{8.8}$$

Harke gives the following rule for commuting the operands of the Hestenes inner product in his paper [84, p. 6, Eq. (22)]. For homogeneous multivectors with subscripts indicating their grades

$$A_r \cdot B_s = (-1)^{s(s-r)} B_s \cdot A_r, \ \text{ if } r \geq s. \tag{8.9}$$

Since Harke's derivation of Equation (6.19) also applies to the fat dot inner product, we may substitute it for the Hestenes inner product and write

$$A_r \bullet B_s = (-1)^{s(s-r)} B_s \bullet A_r, \ \text{ if } r \geq s. \tag{8.10}$$

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The next theorem relates the fat dot inner product to the Clifford product.

Theorem 8.4. *Let $\mathbf{A}_r, \mathbf{B}_s \in \mathcal{C}\ell_{p,q}$ be blades written with subscripts indicating their grades. Then by Definition 5.4 each can be written as an orientation congruent multiproduct, with any grouping into binary products, of r or s pairwise anticommuting vectors. In particular, let $\mathbf{B}_s = \mathbf{b}_1 \circledcirc \cdots \circledcirc \mathbf{b}_i \circledcirc \cdots \circledcirc \mathbf{b}_s$ where all $\mathbf{a}_i \in V^n$ and $\mathbf{a}_i \circledcirc \mathbf{a}_j = -\mathbf{a}_j \circledcirc \mathbf{a}_i$ for all $i \neq j$.*

with $\mathbf{A}_r \wedge \mathbf{b}_i = 0$ for all $1 \leq i \leq s$. Therefore $r \geq s$ and

$$\begin{aligned} \mathbf{B}_s \bullet \mathbf{A}_r &= \mathbf{B}_s \circ \mathbf{A}_r \quad \textit{and} \\ \mathbf{A}_r \bullet \mathbf{B}_s &= \mathbf{A}_r \circ \mathbf{B}_s. \end{aligned} \tag{8.11}$$

Proof. The proof follows from, Theorem 7.1, the Fundamental Clifford Product Decomposition Theorem and Definition. □

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Here we will give two definitions, based on Lounesto [123, pp. 288–90], of the four contractions $\{\, \text{left}, \text{right} \,\} \times \{\, \mathcal{C}\ell, \mathcal{OC} \,\}$. See also Dorst [60, p. 8], for another exposition of the first derivation, and Fauser [68, pp. 23 f.] for another version of both. Let Table 8.1 define notations for the four contraction operators.

We assume that we have already made the extension from $V^n \times V^n$ to $\bigwedge V^n \times \bigwedge V^n$ of the bilinear form[61] associated with a general (not necessarily nondegenerate) quadratic form Q, perhaps, by means such as the references cited above employ. A pair of angular brackets $\langle \bullet, \bullet \rangle$ will denote both the original, unextended bilinear form and its extension.

A general contraction operator may be fundamentally defined as the *dual* or *adjoint* of a modified exterior multiplication with respect to some *pairing,*[62] Depending on the modifications made to exterior product this definition produces a different contraction operator. The modifications required to produce a Clifford or orientation congruent, left or right, contraction operator involve only the reversion of some of the terms. The pairing required is the multilinear extension of the bilinear form associated with the quadratic form of the Clifford or orientation congruent algebra.

[61] This concept was introduced earlier by Definition 5.1 under the notation $B_Q(\bullet, \bullet)$.

[62] A pairing, or bilinear form over $\mathbb{R}$, is defined as a bilinear map $B\colon U \times V \to \mathbb{R}$ where U and V are vector spaces over $\mathbb{R}$ [192, p. 58]. See footnote 15 for a definition of bilinearity. An example of such a pairing is the *scalar product* of multivectors in a Clifford algebra.

The equations in Table 8.2 give *duality definitions* of the four contraction operators using an extended bilinear form based on a nondegenerate Q on V^n. When viewing this and the other tables of this section, please recall that the notation we use for the reversion of a multivector A is a superscript dagger as in $A^\dagger$, and the notation we use for the grade involution of a multivector A is a overlying or superscript circumflex as in $\widehat{A}$ or $A^{\frown}$.

A set of three equations may be derived from the duality definition of each contraction operator; or, conversely, a set of these three equations may be used define the contraction operator that corresponds to it. These sets of three equations may be used to reduce an expression involving the contraction operators to another containing multivectors of lower grade than those in the original expression. Interestingly, these *reduction definitions* are more general than the duality ones; they allow the use of an extended bilinear form that is derived from a general, *possibly degenerate* quadratic form.

The first equation in the set of three is the same for all four operators as is shown in Table 8.3.The other two equations in the set vary by the operator according to Tables 8.4 and 8.5.

8.3. **Derived Expressions for the Contraction Operators.** Lounesto, in his book [123, pp. 38 f.], defines the *Hodge dual* of a multivector for $\mathcal{C}\ell_3$. He writes the Hodge dual of a multivector A using a five-pointed star as a prefix $\star A$, but we write it instead using an asterisk either as a prefix $*A$ or superscript A^*. The operator that produces the Hodge dual of a multivector, $*$, is commonly called the *Hodge star* (or, simply, the *Hodge*) operator.

Lounesto's definition, although stated for $\mathcal{C}\ell_3$, is immediately generalizable to $\mathcal{C}\ell_{p,q}$ because it can be straightforwardly seen to be equivalent to the fourth equation on page 166 of Burke's book [34] (but with multivectors substituted for differential forms). Therefore, we give the following general definition of the Hodge dual operator. For all $A, B \in \mathcal{C}\ell_{p,q}$ and for all extended bilinear forms $\langle \bullet, \bullet \rangle$ derived from a general, possibly degenerate quadratic form

$$A \wedge *B = B \wedge *A = \langle A, B\rangle \mathbf{I}. \tag{8.12}$$

Lounesto [123, p. 39, fn. 6] states, while Benn and Tucker [18, p. 28] derive, an equivalent expression for the Hodge dual of a multivector in terms of the Clifford product:

$$* A = A^* = A^\dagger \circ \mathbf{I} = A^{\dagger \mathbf{I}}. \tag{8.13}$$

The Hodge dual may also be written in terms of the Clifford contraction operators as

$$* A = A^* = A^\dagger \lrcorner \mathbf{I} = (\mathbf{I}^\dagger \llcorner A)^\dagger. \tag{8.14}$$

For an r-blade $\mathbf{A}$, the Hodge dual is also given by

$$\begin{aligned} *\mathbf{A} &= (-1)^{\frac{1}{2}r(r-1)} \mathbf{A} \lrcorner \mathbf{I} \\ &= (-1)^{\frac{1}{2}n(n-1)} (\mathbf{I} \llcorner \mathbf{A})^\dagger \\ &= (-1)^{\frac{1}{2}(n-r)(n-r-1)} \mathbf{I}^\dagger \llcorner \mathbf{A} \\ &= (-1)^{nr+\frac{1}{2}r(r+1)} \mathbf{I} \llcorner \mathbf{A}. \end{aligned}$$

Employing these expressions for the Hodge dual in terms of the Clifford product or contraction operators and some other results to be added to a later version of this paper we may derive the equivalent expressions for the contraction operators given in Table 8.6.

TO BE DEVELOPED FURTHER

TABLE 8.1. Notations for the Four Contraction Operators.

Algebra	Left	Right
$\mathcal{C}\ell$	$\lrcorner$	$\llcorner$
$\mathcal{OC}$	$\neg$	$\ulcorner$

TABLE 8.2. Duality Definitions of the Four Contraction Operators. These definitions are valid for all $u, v, w \in \bigwedge V^n$, and for all extended bilinear forms $\langle \bullet, \bullet \rangle$ derived from a nondegenerate quadratic form.

Algebra	Left Contraction	Right Contraction
$\mathcal{C}\ell$	$\langle u \lrcorner v, w\rangle := \langle v, u^\dagger \wedge w\rangle$	$\langle u \llcorner v, w\rangle := \langle u, w \wedge v^\dagger\rangle$
$\mathcal{OC}$	$\langle u \neg v, w\rangle := \langle v, w \wedge u\rangle$	$\langle u \ulcorner v, w\rangle := \langle u, v \wedge w\rangle$

TABLE 8.3. Reduction Definitions of the Four Contraction Operators: Part 1. These definitions are valid for all $\mathbf{x}, \mathbf{y} \in V^n$, and for all extended bilinear forms $\langle \bullet, \bullet \rangle$ derived from a general, possibly degenerate quadratic form.

Algebra	Left Contraction	Right Contraction
$\mathcal{C}\ell$	$\mathbf{x} \lrcorner \mathbf{y} := \langle \mathbf{x}, \mathbf{y}\rangle$	$\mathbf{x} \llcorner \mathbf{y} := \langle \mathbf{x}, \mathbf{y}\rangle$
$\mathcal{OC}$	$\mathbf{x} \neg \mathbf{y} := \langle \mathbf{x}, \mathbf{y}\rangle$	$\mathbf{x} \ulcorner \mathbf{y} := \langle \mathbf{x}, \mathbf{y}\rangle$

TABLE 8.4. Reduction Definitions of the Four Contraction Operators: Part 2. These definitions are valid for all $\mathbf{x} \in V^n$, for all $u, v \in \bigwedge V^n$, and for all extended bilinear forms $\langle \bullet, \bullet \rangle$ derived from a general, possibly degenerate quadratic form.

Algebra	Left Contraction
$\mathcal{C}\ell$	$\mathbf{x} \lrcorner (u \wedge v) := (\mathbf{x} \lrcorner u) \wedge v + \widehat{u} \wedge (\mathbf{x} \lrcorner v)$
$\mathcal{OC}$	$\mathbf{x} \neg (u \wedge v) := u \wedge (\mathbf{x} \neg v) + (\mathbf{x} \neg u) \wedge \widehat{v}$

Algebra	Right Contraction
$\mathcal{C}\ell$	$(u \wedge v) \llcorner \mathbf{x} := u \wedge (v \llcorner \mathbf{x}) + (u \llcorner \mathbf{x}) \wedge \widehat{v}$
$\mathcal{OC}$	$(u \wedge v) \ulcorner \mathbf{x} := (u \ulcorner \mathbf{x}) \wedge v + \widehat{u} \wedge (v \ulcorner \mathbf{x})$

Table 8.5. Reduction Definitions of the Four Contraction Operators: Part 3. These definitions are valid for all $u, v, w \in \bigwedge V^n$, and for all extended bilinear forms $\langle \bullet, \bullet \rangle$ derived from a general, possibly degenerate quadratic form.

Algebra	Left Contraction	Right Contraction
$\mathcal{C}\ell$	$(u \wedge v) \lrcorner w := u \lrcorner (v \lrcorner w)$	$w \llcorner (u \wedge v) := (w \llcorner u) \llcorner v$
$\mathcal{OC}$	$(u \wedge v) \urcorner w := u \urcorner (v \lrcorner w)$	$w \ulcorner (u \wedge v) := (w \ulcorner u) \ulcorner v$

Table 8.6. Derived Expressions for the Four Contraction Operators. These expressions are valid for all $u, v \in \bigwedge V^n$. The star $*$ represents the *Hodge dual* operator (see the text for details).

	Algebra	Left Contraction	Right Contraction
(1)		$u \lrcorner v$	$u \llcorner v$
(2)		$(v^\dagger \llcorner u^\dagger)^\dagger$	$(v^\dagger \lrcorner u^\dagger)^\dagger$
(3)		$(u \urcorner v^\dagger)^\dagger$	$(u^\dagger \ulcorner v)^\dagger$
(4)	$\mathcal{C}\ell$	$v \ulcorner u^\dagger$	$v^\dagger \urcorner u$
(5)		$[u \wedge (v \circ \boldsymbol{I})] \circ \boldsymbol{I}^{-1}$	$\boldsymbol{I}^{-1} \circ [(\boldsymbol{I} \circ u) \wedge v]$
(6)		$\{\boldsymbol{I}^{-1} \circ [(\boldsymbol{I} \circ v^\dagger) \wedge u^\dagger]\}^\dagger$	$\{[v^\dagger \wedge (u^\dagger \circ \boldsymbol{I})] \circ \boldsymbol{I}^{-1}\}^\dagger$
(7)		$\{*^{-1}[u \wedge *(v^\dagger)]\}^\dagger$	$\{*[*^{-1}(u^\dagger) \wedge v]\}^\dagger$
(8)		$*[(*^{-1}v) \wedge u^\dagger]$	$*^{-1}[v^\dagger \wedge (*u)]$
(9)		$u \urcorner v$	$u \ulcorner v$
(10)		$(v^\dagger \ulcorner u^\dagger)^\dagger$	$(v^\dagger \urcorner u^\dagger)^\dagger$
(11)		$(u \lrcorner v^\dagger)^\dagger$	$(u^\dagger \llcorner v)^\dagger$
(12)	$\mathcal{OC}$	$v \llcorner u^\dagger$	$v^\dagger \lrcorner u$
(13)		$\{[u \wedge (v^\dagger \circ \boldsymbol{I})] \circ \boldsymbol{I}^{-1}\}^\dagger$	$\{\boldsymbol{I}^{-1} \circ [(\boldsymbol{I} \circ u^\dagger) \wedge v]\}^\dagger$
(14)		$\boldsymbol{I}^{-1} \circ [(\boldsymbol{I} \circ v) \wedge u^\dagger]$	$[v^\dagger \wedge (u \circ \boldsymbol{I})] \circ \boldsymbol{I}^{-1}$
(15)		$*^{-1}[u \wedge (*v)]$	$*[(*^{-1}u) \wedge v]$
(16)		$\{*[*^{-1}(v^\dagger) \wedge u^\dagger]\}^\dagger$	$\{*^{-1}[v^\dagger \wedge *(u^\dagger)]\}^\dagger$

9. Some Important Formulas

> The work of Gibbs and then Dirac would have been considerably simplified had Gibbs and Dirac even a fleeting acquaintance with exterior algebra. Instead, exterior algebra ended up another missed opportunity, as Freeman Dyson might say.
>
> Gian-Carlo Rota [152, p. 47]

Some introductory text at the beginning. More introductory text at the beginning.

Citation test: *Geometric Algebra for Computer Science* by Dorst, Fontijne, and Mann [63], *Differential Geometry and Topology: With a View to Dynamical Systems* by Burns and Gidea [37], *Lecture Notes on Elementary Topology and Geometry* by Singer and Thorpe [168], *The Design of Linear Algebra and Geometry* by Hestenes [94], *Projective Geometry with Clifford algebra* by Hestenes and Ziegler [98], *The Dirac-Kähler Equation and Fermions on the Lattice* by Becher and Joos [12], *Smarandache Non-Associative rings* by Vasantha Kandasamy [189] .

9.1. **The Null Associator Predictor.** As is standard, we use the term *nonassociative* in the sense "not necessarily associative." Thus, the set of nonassociative algebras includes the associative ones as well as the algebras that are not associative. Also, we use the word *algebra* to mean *nonassociative algebra.* We indicate the product operation of a general algebra $\mathfrak{A}$ by juxtaposition.

Definition 9.1. In the theory of nonassociative algebras the *associator* of any three elements, u, v, w, of an algebra $\mathfrak{A}$ is usually written with parentheses as (u, v, w) [156, p. 13] or with square brackets as $[u, v, w]$ [102, p. 435]. However, we prefer to write the associator in operator notation as $\operatorname{asc}(u, v, w)$. The associator is a trilinear function [102, p. 435] that is defined by

$$\operatorname{asc}(u, v, w) := (uv)w - u(vw),$$

for all $u, v, w \in \mathfrak{A}$.

In any algebra $\mathfrak{A}$, the following statements are equivalent:

- The algebra $\mathfrak{A}$ is associative.
- All triples in $\mathfrak{A}$ are associative.
- The associator is null, $\operatorname{asc}(u, v, w) = 0$, for all $(u, v, w) \in \mathfrak{A}^3$.

9.2. **Groups, Loops, and Algebras.** Hamilton's quaternions form an algebra $\mathbb{H}$ which is associative, but not commutative. It can be defined from the vector space of all elements $q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where $w, x, y, z \in \mathbb{R}$, by taking 1 as the identity element and enforcing these equalities: $\mathbf{i}\,\mathbf{j} = -\mathbf{j}\,\mathbf{i} = \mathbf{k}$, $\mathbf{j}\,\mathbf{k} = -\mathbf{k}\,\mathbf{j} = \mathbf{i}$, $\mathbf{k}\,\mathbf{i} = -\mathbf{i}\,\mathbf{k} = \mathbf{j}$, and $\mathbf{i}\,\mathbf{i} = \mathbf{j}\,\mathbf{j} = \mathbf{k}\,\mathbf{k} = -1$. We can derive a couple of groups from this algebra by defining the group operation to be quaternion multiplication and the group set to be a particular subset of $\mathbb{H}$.

Example 9.2. By restricting the set of elements of $\mathbb{H}$ to the set $\pm\mathscr{B}^1_{\mathbb{H}} = \pm(\{1\} \cup \mathscr{B}_{\mathbb{H}})$, which contains the elements, and their negatives, in the union of the set $\{1\}$ containing the identity and the set $\mathscr{B}_{\mathbb{H}}$ of standard basis elements of $\mathbb{H}$, we obtain the order eight *quaternion frame group* $\mathbf{\Gamma}(\mathbb{H}, \pm\mathscr{B}^1_{\mathbb{H}})$. Thus, the following set comprises the elements of the quaternion frame group: $\mathbf{\Gamma}(\mathbb{H}, \pm\mathscr{B}^1_{\mathbb{H}}) = \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$.

The *quaternion line group* $\mathbf{\Gamma}(\mathbb{H}, \mathbb{R}^{\bullet}\mathscr{B}^1_{\mathbb{H}})$ is also derived from Hamilton's quaternion algebra $\mathbb{H}$.

Example 9.3. By restricting the set of elements of $\mathbb{H}$ to the set $\mathbb{R}^{\bullet}\mathscr{B}^1_{\mathbb{H}}$ of all nonzero real multiples of the set $\mathscr{B}^1_{\mathbb{H}}$ defined above in Example 9.2, we obtain the infinite *quaternion line group* $\mathbf{\Gamma}(\mathbb{H}, \mathbb{R}^{\bullet}\mathscr{B}^1_{\mathbb{H}})$. Thus, the following set comprises the elements of the quaternion line group: $\mathbf{\Gamma}(\mathbb{H}, \mathbb{R}^{\bullet}\mathscr{B}^1_{\mathbb{H}}) = \{\alpha, \alpha\mathbf{i}, \alpha\mathbf{j}, \alpha\mathbf{k} \mid \alpha \in \mathbb{R}^{\bullet} = \mathbb{R} \setminus \{0\}\}$.

Example 9.4. By restricting the set of elements of the Clifford algebra $\mathcal{C}\ell_{p,q}$ to the set $\mathbb{R}^{\bullet}\mathscr{B}^{\wedge}_{\mathcal{C}\ell_{p,q}}$ of all nonzero real multiples of its standard basis blades, we obtain the infinite *Clifford line group* $\mathbf{\Gamma}(\mathcal{C}\ell_{p,q}, \mathbb{R}^{\bullet}\mathscr{B}^{\wedge}_{\mathcal{C}\ell_{p,q}})$. Thus, the following set comprises the elements of the Clifford line group: $\mathbf{\Gamma}(\mathcal{C}\ell_{p,q}, \mathbb{R}^{\bullet}\mathscr{B}^{\wedge}_{\mathcal{C}\ell_{p,q}}) = \{\alpha, \alpha\mathbf{e}_1, \alpha\mathbf{e}_2, \ldots, \alpha\mathbf{e}_n, \alpha\mathbf{e}_{12}, \alpha\mathbf{e}_{13}, \ldots, \alpha\mathbf{e}_{12\ldots n} \mid \alpha \in \mathbb{R}^{\bullet} = \mathbb{R} \setminus \{0\}\}$.

Example 9.5. The *Clifford blade group operator* $\overline{\circ}$ is also known as the *(greater) delta product* Δ [14, 26, 27]. By restricting the set of elements of the Clifford algebra $\mathcal{C}\ell_n$ to the set $\mathcal{B}\ell_{\mathcal{C}\ell_n}$ of all its blades, we obtain the infinite *Clifford blade group* $\mathbf{\Gamma}(\mathcal{C}\ell_n, \mathcal{B}\ell_{\mathcal{C}\ell_n}, \overline{\circ})$. Thus, the following set comprises the elements of the Clifford blade group: $\mathbf{\Gamma}(\mathcal{C}\ell_n, \mathcal{B}\ell_{\mathcal{C}\ell_n}, \overline{\circ}) = \{\alpha, \mathbf{v}_1, \mathbf{v}_1 \circ \mathbf{v}_2 \circ \cdots \circ \mathbf{v}_i \circ \cdots \mathbf{v}_r \mid \alpha \in \mathbb{R}^{\bullet},\ \mathbf{v}_i \neq 0,\ \mathbf{v}_i \in \langle \mathcal{C}\ell_n \rangle_1,\ r \in \mathbb{Z}[2, n], \text{ and } \mathbf{v}_i \circ \mathbf{v}_j = -\mathbf{v}_j \circ \mathbf{v}_i \text{ for all } i \neq j\}$.

MacFarlane's hyperbolic quaternions form an algebra $\mathbb{M}$ which is neither associative nor commutative [125, 126, 200]. It can be defined from the vector space of all elements $q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where $w, x, y, z \in \mathbb{R}$, by taking 1 as the identity element and enforcing these equalities: $\mathbf{i}\,\mathbf{j} = -\mathbf{j}\,\mathbf{i} = \mathbf{k}$, $\mathbf{j}\,\mathbf{k} = -\mathbf{k}\,\mathbf{j} = \mathbf{i}$, $\mathbf{k}\,\mathbf{i} = -\mathbf{i}\,\mathbf{k} = \mathbf{j}$, and $\mathbf{i}\,\mathbf{i} = \mathbf{j}\,\mathbf{j} = \mathbf{k}\,\mathbf{k} = 1$. The crucial difference from Hamilton's quaternion algebra is the positive sign of 1 in the final chain of equalities. We can derive a couple of loops from this algebra by defining the loop operation to be hyperbolic quaternion multiplication and the group set to be a particular subset of $\mathbb{M}$.

Example 9.6. By restricting the set of elements of $\mathbb{M}$ to the set $\pm\mathscr{B}^1_{\mathbb{M}} = \pm(\{1\} \cup \mathscr{B}_{\mathbb{M}})$, which contains the elements, and their negatives, in the union of the set $\{1\}$ containing the identity and the set $\mathscr{B}_{\mathbb{M}}$ of standard basis elements of $\mathbb{M}$, we obtain the order eight *hyperbolic quaternion frame loop* $\mathbf{\Lambda}(\mathbb{M}, \pm\mathscr{B}^1_{\mathbb{M}})$. Thus, the following set comprises the elements of the hyperbolic quaternion frame loop: $\mathbf{\Lambda}(\mathbb{M}, \pm\mathscr{B}^1_{\mathbb{M}}) = \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$.

The *hyperbolic quaternion line loop* $\mathbf{\Lambda}(\mathbb{M}, \mathbb{R}^{\bullet}\mathscr{B}^1_{\mathbb{M}})$ is also derived from MacFarlane's hyperbolic quaternion algebra $\mathbb{M}$.

Example 9.7. By restricting the set of elements of $\mathbb{M}$ to the set $\mathbb{R}^{\bullet}\mathscr{B}^{1}_{\mathbb{M}}$ of all nonzero real multiples of the set $\mathscr{B}^{1}_{\mathbb{M}}$ defined above in Example 9.6, we obtain the infinite *hyperbolic quaternion line loop* $\mathbf{\Lambda}(\mathbb{M}, \mathbb{R}^{\bullet}\mathscr{B}^{1}_{\mathbb{M}})$. Thus, the following set comprises the elements of the hyperbolic quaternion line loop: $\mathbf{\Lambda}(\mathbb{M}, \mathbb{R}^{\bullet}\mathscr{B}^{1}_{\mathbb{M}}) = \{\alpha, \alpha\mathbf{i}, \alpha\mathbf{j}, \alpha\mathbf{k} \mid \alpha \in \mathbb{R}^{\bullet} = \mathbb{R} \setminus \{0\}\}$.

Example 9.8. By restricting the set of elements of the orientation congruent algebra $\mathcal{OC}_{p,q}$ to the set $\mathbb{R}^{\bullet}\mathscr{B}^{\wedge}_{\mathcal{OC}_{p,q}}$ of all nonzero real multiples of its standard basis blades, we obtain the infinite *orientation congruent line loop* $\mathbf{\Lambda}(\mathcal{OC}_{p,q}, \mathbb{R}^{\bullet}\mathscr{B}^{\wedge}_{\mathcal{OC}_{p,q}})$. Thus, the following set comprises the elements of the orientation congruent line loop: $\mathbf{\Lambda}(\mathcal{OC}_{p,q}, \mathbb{R}^{\bullet}\mathscr{B}^{\wedge}_{\mathcal{OC}_{p,q}}) = \{\alpha, \alpha\mathbf{e}_1, \alpha\mathbf{e}_2, \ldots, \alpha\mathbf{e}_n, \alpha\mathbf{e}_{12}, \alpha\mathbf{e}_{13}, \ldots, \alpha\mathbf{e}_{12\ldots n} \mid \alpha \in \mathbb{R}^{\bullet} = \mathbb{R} \setminus \{0\}\}$.

The algebra $\mathcal{B}\ell$, with subscript $\mathcal{B}\ell_{p,q}$. blade algebra
The algebra $\boldsymbol{\mathcal{B}\ell}$, with subscript $\boldsymbol{\mathcal{B}\ell_{p,q}}$. blade algebra BOLD
The algebra $\mathcal{OB}\ell$, with subscript $\mathcal{OB}\ell_{p,q}$. oriented blade algebra
The algebra $\boldsymbol{\mathcal{OB}\ell}$, with subscript $\boldsymbol{\mathcal{OB}\ell_{p,q}}$. oriented blade algebra BOLD
The product $\mathbf{A} \circ \mathbf{B} = \mathbf{C}$. Clifford algebra product
The product $\mathbf{A} \mathbin{\bar{\circ}} \mathbf{B} = \mathbf{C}$. blade algebra product
The product $\mathbf{A} \circledcirc \mathbf{B} = \mathbf{C}$. orientation congruent algebra product
The product $\mathbf{A} \mathbin{\overline{\circledcirc}} \mathbf{B} = \mathbf{C}$. oriented blade algebra product
Test citation [43, pp. 462–465];
Test citation [28].

We now need another type [concept] of associator that is analogous to the associator of an algebra. We need to discuss the associator of a nonempty set $\mathscr{L} \subseteq \mathfrak{A}$ that is the subset an algebra and that is

also a *loop* under the algebra product. A loop is a quasigroup with an identity element.

Definition 9.9. In the theory of nonassociative algebras the *associator* of any three elements, u, v, w, of an algebra $\mathfrak{A}$ is usually written with a pair of enclosing parentheses as (u, v, w) [156, p. 13]. However, we prefer to keep parentheses for ordered n-tuples and so we write it the associator in operator notation as $\operatorname{asc}(u, v, w)$. The associator is defined by

$$\operatorname{asc}(u, v, w) := (uv)w - u(vw),$$

where the product of $\mathfrak{A}$ is written as juxtaposition.

The associator is a multilinear operator. Therefore, $\operatorname{asc}(u, v, w) = (1 \pm 1)(uv)w$. This prompts the following definition.

Definition 9.10. For any algebra $\mathfrak{A}$ and any set $\mathscr{L}$ such that $\varnothing \subsetneq \mathscr{L} \subseteq \mathfrak{A}$, that is a loop (a quasigroup with an identity) under the algebra product (but not necessarily a subalgebra) we call any function $\operatorname{nap}\colon \mathscr{L}^3 \to \mathbb{Z}$, from the set of triples (u, v, w) of elements in $\mathscr{L}$ to the integers, a *null associator predictor for* $\mathscr{L}$ if and only if

$$\operatorname{asc}(u, v, w) = \left(1 - (-1)^{\operatorname{nap}(u,v,w)}\right)(uv)w$$

for all $(u, v, w) \in \mathscr{L}^3$.

Some examples of a set $\mathscr{L} \subseteq \mathfrak{A}$ that is a loop under the algebra product include, $\pm\mathscr{B}^{\wedge}$, the negative extension of some set of basis blades for the orientation congruent algebra $\mathcal{OC}_{p,q}$, or, $\mathscr{D}$, the set of all *decomposable* multivectors or *blades* in $\mathcal{OC}_{p,q}$.

In any algebra $\mathfrak{A}$ the associator of a triple $\operatorname{asc}(u, v, w)$ is null if and only if $\operatorname{nap}_1(u, v, w)$ is even for some null associator predictor nap_1. Two null associator predictors for $\mathscr{L}$, nap_1 and nap_2, are *equivalent* if and only if $\operatorname{nap}_1(u, v, w) \equiv \operatorname{nap}_2(u, v, w) \mod 2$ for all $(u, v, w) \in \mathscr{L}^3$.

Generally, in the sequel we employ the notation asc to symbolize the associator exclusively for the orientation congruent algebra. Also, in the sequel we employ the notation nap for a specific null associator predictor, that is defined below, for the $\mathcal{OC}$ algebra, namely the *standard null associator predictor*.

Definition 9.11. Let $\mathscr{D}$ be the set of *decomposable* multivectors or *blades* and let $\mathbb{N}$ be the set of natural numbers including 0. Then the symbol $\gamma(\bullet)$ represents the *grade of* operator $\gamma\colon \mathscr{D} \to \mathbb{N}$ which gives the degree of a decomposable multivector in $\bigwedge V^n$.

Definition 9.12. The symbol $\cap$ represents the *meet* operator.

Theorem 9.13. *In an orientation congruent algebra $\mathcal{OC}_{p,q}$ a triple of blades $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ has a null associator if and only if the following* null associator predictor *is even:*

$$\Big(\gamma(A \cap C) + \gamma(A)\,\gamma(C)\Big)\Big(\gamma(A \cap B) + \gamma(B \cap C)\Big) + \gamma(A \cap C)\,\gamma(B)\Big(\gamma(A) + \gamma(C)\Big). \tag{9.1}$$

Proof. A lot of tedious algebraic manipulation of sign factor functions that is left to the reader or the reader's symbolic computer algebra system. □

Theorem 9.14. *In an orientation congruent algebra $\mathcal{OC}_{p,q}$ the following expression is the restriction the* standard null associator predictor *to all triples, (A_r, B_s, C_t), of homogeneous multivectors with subscripts indicating their grades we obtain this equation:*

$$\Big(\gamma(A \cap C) + \gamma(A)\,\gamma(C)\Big)\Big(\gamma(A \cap B) + \gamma(B \cap C)\Big) + \gamma(A \cap C)\,\gamma(B)\Big(\gamma(A) + \gamma(C)\Big). \tag{9.2}$$

Proof. A lot of tedious algebraic manipulation of sign factor functions that is left to the reader or the reader's symbolic computer algebra system. □

Note the symmetry in this expression. The A and C at the extreme positions of the associator always appear in balanced pairs that commute.

9.3. **Indeterminate Counit Equations.** Some introductory remarks.

Theorem 9.15. *For all blades $\mathbf{A} \in \mathcal{C}\ell_{p,q}$*

$$\mathbf{A} \circ \mathbf{I} = \mathbf{A} \,\lrcorner\, \mathbf{I} \quad \textit{and} \tag{9.3a}$$

$$\mathbf{I} \circ \mathbf{A} = \mathbf{I} \,\llcorner\, \mathbf{A}. \tag{9.3b}$$

Proof. We prove only Equation (9.3a), the second equation follows by symmetry.

If $\mathbf{A}$ is the scalar α, Equation (9.3a) follows immediately from the definition of the left Clifford contraction in Equation (8.4) of Definition 8.2 of Section 8.

Now assume $\mathbf{A}$ is an r-blade for $1 \leq r \leq n$. Since both $\mathbf{A}$ and $\mathbf{I}$ are blades we may factor them into Clifford products of mutually anticommuting vectors. We first factor $\mathbf{A}$ into r such vectors: $\mathbf{A} = \mathbf{a}_r \circ \cdots \circ \mathbf{a}_2 \circ \mathbf{a}_1$. Then we factor $\mathbf{I}$ so that its first r vectors are the same as those for $\mathbf{A}$, but in reversed order: $\mathbf{I} = \mathbf{a}_1 \circ \mathbf{a}_2 \circ \ldots \circ \mathbf{a}_r \circ \mathbf{b}_1 \circ \ldots \circ \mathbf{b}_{n-r-1} \circ \mathbf{b}_{n-r}$. If $\mathbf{A} = \alpha\mathbf{I}$ for some scalar α, we must include this scalar in the factorization of $\mathbf{A}$. However, to simplify the following derivations we do not include α.

Using these factorizations we obtain the following expression for the left side of Equation (9.3a)

$$\mathbf{A} \circ \mathbf{I} = (\mathbf{a}_r \circ \cdots \circ \mathbf{a}_2 \circ \mathbf{a}_1) \circ (\mathbf{a}_1 \circ \mathbf{a}_2 \circ \ldots \circ \mathbf{a}_r \circ \mathbf{b}_1 \circ \ldots \circ \mathbf{b}_{n-r-1} \circ \mathbf{b}_{n-r}).$$

Now we can successively "consume" pairs of vectors at the junction of these decompositions of $\mathbf{A}$ and $\mathbf{I}$ by repeatedly Clifford multiplying them to yield scalars. Eventually the left side of Equation (9.3a) is reduced to $(\mathbf{a}_1^{\circ 2}\mathbf{a}_2^{\circ 2} \cdots \mathbf{a}_r^{\circ 2})\mathbf{b}_1 \circ \ldots \circ \mathbf{b}_{n-r-1} \circ \mathbf{b}_{n-r}$.

Next, we work on the right side of Equation (9.3a). Since the vectors in the decomposition of $\mathbf{A}$ anticommute we may replace the Clifford products in this decomposition with outer products to yield: $\mathbf{A} = \mathbf{a}_r \wedge \cdots \wedge \mathbf{a}_2 \wedge \mathbf{a}_1$. The same transformation applies to $\mathbf{I}$. By repeatedly applying the expression from Table 8.5 in Section 8 for the left Clifford contraction of an outer product with a multivector and then the expression from Table 8.4 in Section 8 for the left Clifford contraction of a vector with an outer product we generate the following series of equivalent expressions:

$$\begin{aligned}
\mathbf{A} \rfloor \mathbf{I} &= (\mathbf{a}_r \wedge \cdots \wedge \mathbf{a}_3 \wedge \mathbf{a}_2 \wedge \mathbf{a}_1) \rfloor (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3 \wedge \ldots \wedge \mathbf{a}_r \wedge \mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_{n-r}) \\
&= (\mathbf{a}_r \wedge \cdots \wedge \mathbf{a}_3 \wedge \mathbf{a}_2) \rfloor [\mathbf{a}_1 \rfloor (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3 \wedge \ldots \wedge \mathbf{a}_r \wedge \mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_{n-r})] \\
&= (\mathbf{a}_r \wedge \cdots \wedge \mathbf{a}_3 \wedge \mathbf{a}_2) \rfloor [\mathbf{a}_1^{\rfloor 2}(\mathbf{a}_2 \wedge \mathbf{a}_3 \wedge \ldots \wedge \mathbf{a}_r \wedge \mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_{n-r})] \\
&= (\mathbf{a}_r \wedge \cdots \wedge \mathbf{a}_3) \rfloor \{\mathbf{a}_2 \rfloor [\mathbf{a}_1^{\rfloor 2}(\mathbf{a}_2 \wedge \mathbf{a}_3 \wedge \ldots \wedge \mathbf{a}_r \wedge \mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_{n-r})]\} \\
&= (\mathbf{a}_r \wedge \cdots \wedge \mathbf{a}_3) \rfloor [(\mathbf{a}_1^{\rfloor 2}\mathbf{a}_2^{\rfloor 2})(\mathbf{a}_3 \wedge \ldots \wedge \mathbf{a}_r \wedge \mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_{n-r})] \\
&\;\;\vdots \\
&= (\mathbf{a}_1^{\rfloor 2}\mathbf{a}_2^{\rfloor 2} \cdots \mathbf{a}_r^{\rfloor 2})(\mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_{n-r}).
\end{aligned}$$

Since, for any vector $\mathbf{a}$, $\mathbf{a}^{\circ 2} := \mathbf{a} \circ \mathbf{a}$, $\mathbf{a}^{\rfloor 2} := \mathbf{a} \rfloor \mathbf{a}$, and $\mathbf{a} \circ \mathbf{a} = \mathbf{a} \rfloor \mathbf{a}$ by Equation (8.4) of Definition 8.2 in Section 8, the last expression in this series is equal to the result above for the left side of Equation (9.3a). $\square$

Theorem 9.16. *For all blades* $\mathbf{A} \in \mathcal{C}\ell_{p,q}$

$$\mathbf{A} \circ \mathbf{I} = \mathbf{A} \rfloor \mathbf{I} = \mathbf{A} \bullet \mathbf{I} \quad \textit{and} \tag{9.4a}$$

$$\mathbf{I} \circ \mathbf{A} = \mathbf{I} \llcorner \mathbf{A} = \mathbf{I} \bullet \mathbf{A}. \tag{9.4b}$$

Proof. We prove only Equation (9.4a), the second equation follows by symmetry. The equality of the Clifford product and the left Clifford contraction was proved in Theorem 9.15 above. The equality of the left Clifford contraction and the Clifford fat dot inner product follows from their definitions by Equations (8.4) and (8.3) of Definition 8.2 in Section 8. $\square$

We immediately have the following theorem as the orientation congruent analogue of Theorem 9.16.

Theorem 9.17. *For all blades* $\mathbf{A} \in \mathcal{OC}_{p,q}$

$$\mathbf{A} \circledcirc \mathbf{\Omega} = \mathbf{A} \urcorner \mathbf{\Omega} = \mathbf{A} \odot \mathbf{\Omega} \quad \textit{and} \tag{9.5a}$$

$$\mathbf{\Omega} \circledcirc \mathbf{A} = \mathbf{\Omega} \ulcorner \mathbf{A} = \mathbf{\Omega} \odot \mathbf{A}. \tag{9.5b}$$

Proof. These equations follow from substituting the sigma definitions of the orientation congruent product, the left and right orientation congruent contractions, and the orientation congruent fat dot inner product in Theorem 9.16 above. $\square$

Theorem 9.18. *For all basis blades* $\mathbf{e}_I \in \mathscr{B}^r$

$${}^{\mathbf{\Omega}}\mathbf{e}_I = (-1)^{r(n-r)}\,\mathbf{e}_I^{\mathbf{\Omega}}. \tag{9.6}$$

Proof.

$$\begin{aligned}
{}^{\mathbf{\Omega}}\mathbf{e}_I &= \mathbf{\Omega} \circledcirc \mathbf{e}_I && \text{by Definitions 5.15 and 5.18}\\
&= \sigma(\mathbf{\Omega}, \mathbf{e}_I)\,\mathbf{\Omega} \circ \mathbf{e}_I && \text{by Definition 6.5}\\
&= \sigma(\mathbf{\Omega}, \mathbf{e}_I)\,\mathbf{\Omega} \bullet \mathbf{e}_I && \text{since the grade of } \mathbf{\Omega} \text{ is } n\\
&= \sigma(\mathbf{\Omega}, \mathbf{e}_I)\,(-1)^{r(n-r)}\,\mathbf{e}_I \bullet \mathbf{\Omega} && \text{by Harke's Eq. (22) [84, p. 6] applied to } \bullet\\
&= \sigma(\mathbf{\Omega}, \mathbf{e}_I)\,(-1)^{r(n-r)}\,\mathbf{e}_I \circ \mathbf{\Omega} && \text{since the grade of } \mathbf{\Omega} \text{ is } n\\
&= (-1)^{r(n-r)}\,\mathbf{e}_I \circledcirc \mathbf{\Omega} && \text{by Definition 6.5}\\
&= (-1)^{r(n-r)}\,\mathbf{e}_I^{\mathbf{\Omega}} && \text{by Definitions 5.15 and 5.18.}
\end{aligned}$$

$\square$

Corollary 9.19. *For all basis blades* $\mathbf{e}_I \in \mathscr{B}^r$

$${}^{\mho}\mathbf{e}_I = (-1)^r\,\mathbf{e}_I^{\mho} = \hat{\mathbf{e}}_I^{\mho}. \tag{9.7}$$

Proof. The proof follows from applying Definitions 5.15 and 5.18 and the definition of grade involution to Theorem 9.18. $\square$

Theorem 9.20. *For any orientation congruent algebra* $\mathcal{OC}_{p,q}$ *and all basis blades* $\mathbf{e}_I \in \mathscr{B}^r \subset \mathcal{OC}_{p,q}$

$$\left({}^{\mathbf{\Omega}}\mathbf{e}_I\right)^{\mathbf{\Omega}} = (-1)^q \mathbf{e}_I = {}^{\mathbf{\Omega}}\!\left(\mathbf{e}_I^{\mathbf{\Omega}}\right). \tag{9.8}$$

Proof. We only prove the equality of the left and middle expressions; the equality of the middle and right expressions follows by symmetry.

$$\begin{aligned}
\left({}^{\mathbf{\Omega}}\mathbf{e}_I\right)^{\mathbf{\Omega}} &= (\mathbf{\Omega} \circledcirc \mathbf{e}_I) \circledcirc \mathbf{\Omega} && \text{by Definitions 5.15 and 5.18}\\
&= (-1)^{r(n-r)}\,(\mathbf{e}_I \circledcirc \mathbf{\Omega}) \circledcirc \mathbf{\Omega} && \text{by Theorem 9.18}
\end{aligned}$$

By Theorem 9.14 the null associator predictor for the triple $(\mathbf{e}_I, \overline{\underline{\mathbf{O}}}, \overline{\underline{\mathbf{O}}})$ is $(r + nr)(r+n) + rn(r+n) \equiv r(n-r) \mod 2$. Therefore, continuing the above series of equalities by shifting the parentheses to the right side of the last expression we have

$$
\begin{aligned}
&= \mathbf{e}_I \circledcirc (\overline{\underline{\mathbf{O}}} \circledcirc \overline{\underline{\mathbf{O}}}) && \text{mod 2 addition of the exponents of } -1 \\
&= (-1)^q \mathbf{e}_I && \text{by Theorem VII.2}'.
\end{aligned}
$$

$\square$

10. Electromagnetic Field Jump Conditions

> One cannot escape the feeling that these equations have an existence and intelligence of their own; that they are wiser than we are, wiser even than their discoverers; that we get more out of them than was originally put into them.
>
> Heinrich Hertz on Maxwell's equations, quoted by E.T. Bell [13, p. 16]

The usual Gibbs-Heaviside vector calculus formulation of the boundary conditions of the electromagnetic fields is given by

$$\begin{aligned} \mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) &= 0 \\ \mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) &= \mathbf{J}_s \end{aligned} \tag{10.1a}$$

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) &= 0 \\ \mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) &= \sigma_s. \end{aligned} \tag{10.1b}$$

Standard derivations of these results are provided in the books of Jackson [101, pp. 17–22]) and Kong [113, pp. 25–29]. A more general derivation is given in the book of Ingarden and Jamiołkowski [100, p. 74].

10.1. **Boundary Conditions with Clifford Algebra.**

Jancewicz first restated these electromagnetic boundary conditions using bivectors and exterior algebra (under the geometric algebra name *outer product*) in the paper [103, pp. 183 f.]. Later, Jancewicz applied Clifford algebra to their *ab initio* derivation in his book [104, pp. 81–88]. In both works he also writes the standard vector calculus equations using field vectors decomposed into components that are tangential (subscript t) to the surface and those that are normal to it (subscript n). The following equivalent versions of the vector calculus expressions of Equations (10.1) reproduce Equations (5) of Jancewicz's paper or those on page 87 of his book with slight changes in order and symbols:

$$\begin{aligned} \mathbf{E}_{1t} - \mathbf{E}_{2t} &= 0 \\ \mathbf{H}_{1t} - \mathbf{H}_{2t} &= \mathbf{J}_s \times \mathbf{n} \end{aligned} \tag{10.2a}$$

$$\begin{aligned} \mathbf{B}_{1n} - \mathbf{B}_{2n} &= 0 \\ \mathbf{D}_{1n} - \mathbf{D}_{2n} &= \sigma_s \mathbf{n}. \end{aligned} \tag{10.2b}$$

Jancewicz first restated these electromagnetic boundary conditions using bivectors and exterior algebra (under the geometric algebra name *outer product*) in the paper [103, pp. 183 f.]. Later, Jancewicz applied Clifford algebra to their *ab initio* derivation in his book [104, pp. 81–88]. In the next set of equations we apply a bold font, prefixed subscript 2 to indicate a grade 2 even multivector as in the bivectors ${}_{\mathbf{2}}H$ and ${}_{\mathbf{2}}B$. These notations differ from the superior lens, as in $\overset{\frown}{\widehat{H}}$, which Jancewicz used in Reference [103, pp. 183 f.], or the circumflex, as in $\widehat{H}$, which he used in Reference [104, pp. 81–88]. The following Clifford algebra equivalents of the vector calculus expressions of Equations (10.1) reproduce the equations immediately after Equations (5) of Jancewicz's paper or those immediately after Figure 31 on page 87 of his book with slight changes in order and symbols:

$$\begin{aligned} \mathbf{E}_{1t} - \mathbf{E}_{2t} &= 0 \\ {}_{\mathbf{2}}H_{1n} - {}_{\mathbf{2}}H_{2n} &= \mathbf{J}_s \wedge \mathbf{n} \end{aligned} \tag{10.3a}$$

$$\begin{aligned} {}_{\mathbf{2}}B_{1t} - {}_{\mathbf{2}}B_{2t} &= 0 \\ \mathbf{D}_{1n} - \mathbf{D}_{2n} &= \sigma_s \mathbf{n}. \end{aligned} \tag{10.3b}$$

Puska's paper [144, p. 14] gives the same jump conditions in an equivalent Clifford algebra formulation. In the next set of equations the associativity of the Clifford product eliminates many parentheses. We follow our general convention of writing the Clifford product explicitly as $\circ$ using a small open circle in these results of Puska:

$$\begin{aligned} \frac{1}{2}(\mathbf{E}_1 - \mathbf{E}_2) - \frac{1}{2}\mathbf{n} \circ (\mathbf{E}_1 - \mathbf{E}_2) \circ \mathbf{n}^{-1} &= 0 \\ \frac{1}{2}(\mathbf{H}_1 - \mathbf{H}_2) - \frac{1}{2}\mathbf{n} \circ (\mathbf{H}_1 - \mathbf{H}_2) \circ \mathbf{n}^{-1} &= \mathbf{n} \circ \mathbf{J}_s \circ \mathbf{e}_{123} \end{aligned} \tag{10.4a}$$

$$\begin{aligned} \frac{1}{2}(\mathbf{B}_1 - \mathbf{B}_2) + \frac{1}{2}\mathbf{n} \circ (\mathbf{B}_1 - \mathbf{B}_2) \circ \mathbf{n}^{-1} &= 0 \\ \frac{1}{2}(\mathbf{D}_1 - \mathbf{D}_2) + \frac{1}{2}\mathbf{n} \circ (\mathbf{D}_1 - \mathbf{D}_2) \circ \mathbf{n}^{-1} &= \mathbf{n}\sigma_s. \end{aligned} \tag{10.4b}$$

In the following reproduction of more of Puska's results we again prefix a bold subscript 2 to even bivectors whereas he symbolizes them with not bold, upright, uppercase letters. Also we use a superscript dagger for the reversion operation as in $(\mathbf{e}_{123})^\dagger = (\mathbf{e}_1 \circ \mathbf{e}_2 \circ \mathbf{e}_3)^\dagger = \mathbf{e}_3 \circ \mathbf{e}_2 \circ \mathbf{e}_1 = -\mathbf{e}_{123}$ whereas Puska uses a tilde. On page 15 of Puska's paper [144, p. 14] we find his Equations (15) and (16) combining our Equations 10.4 into just two similar ones by introducing the bivectors ${}_{\mathbf{2}}B = \mathbf{B} \circ \mathbf{e}_{123}$ and ${}_{\mathbf{2}}H = \mathbf{H} \circ \mathbf{e}_{123}$:

$$\frac{1}{2}\left(c^{-1}\mathbf{E} + {}_{\mathbf{2}}B\right) - \frac{1}{2}\mathbf{n} \circ \left(c^{-1}\mathbf{E} + {}_{\mathbf{2}}B\right)^\dagger \circ \mathbf{n}^{-1} = 0 \tag{10.5a}$$

$$\frac{1}{2}\left(c^{-1}{}_{\mathbf{2}}H + \mathbf{D}\right) + \frac{1}{2}\mathbf{n} \circ \left(c^{-1}{}_{\mathbf{2}}H + \mathbf{D}\right)^\dagger \circ \mathbf{n}^{-1} = \mathbf{n}\sigma_s - c^{-1}\mathbf{n} \circ \mathbf{J}_s. \tag{10.5b}$$

We have divided equations (10.1) and (10.4) into two subgroups based on the degrees of the differential forms which would represent the field vectors. That is, the first subgroups, equations (10.1a) and (10.4a), involve the field vectors **E** and **H** which are equivalent to 1-forms. On the other hand, the second subgroups, equations (10.1b) and (10.4b), involve the field vectors **B** and **D** which are equivalent to 2-forms.

An important feature of these equations is that their form differs between subgroups. That is, in the vector calculus expressions of equations (10.1) the operator changes from the cross product to the scalar product in moving from the first to the second subgroup. On the other hand, in the Clifford algebra expressions of equations (10.4) no operator changes in moving between subgroups. However, there is a change which is less severe than that of an operator, namely, a sign change.

In contrast to both the vector calculus and Clifford algebra formulations of the jump condition equations the one we present in this Section is completely uniform: no operators, no signs change; only the quantities involved do. Although, these equations are form-invariant in our theory, it is, of course, still true that those involving the field intensities **E** and **B** are naturally homogeneous, while those involving the field quantities **D** and **H** are naturally not so.

10.2. **Boundary Conditions with Odd Forms.** In this Subsection we present a coherent resolution of a dilemma involving the surface boundary conditions for discontinuous odd forms representing source electromagnetic fields. For brevity, we refer to these kinds of boundary conditions by the standard term *jump conditions*. This dilemma was first reported and resolved, but incoherently, by Warnick, Selfridge, and Arnold in their pioneering 1995 paper [195, p. 332, fn.]. Later, in a 2006 paper [194, pp. 162 f.], Warnick and Russer also resolve the same dilemma. The resolution of Reference [194] is different than that of Reference [195]. Unfortunately, that resolution is also not coherent. For more details, also see Warnick's groundbreaking Ph.D. Dissertation [193, pp. 16–18, 94–100]. Let us review the work of these authors.

The concept of an odd differential form may be traced to an analogous tensorial version given by Weyl in his book [198]. Although, the related concept of *outer orientation* appears at least as early as Veblen's pioneering topology book [190, pp. 10, 194], and Veblen and Whitehead's differential geometry book [191, pp. 55 f.]. An *outer orientation* can be given a neat, modern definition in terms of quotient spaces and the ordinary, inner orientation of a vector space. This is described by Shaw in his book [166, p. 78].

Burke, first, in the paper [33], later, in the book [34], and finally, in the two draft papers [35] and [36], became the strongest, most recent advocate for the formulation of electrodynamics using both even (ordinary) and odd (twisted) differential forms. Unfortunately, William Lionel Burke died at age 55 in 1996 from a cervical fracture that he suffered in an automobile accident. See his Wikipedia entry [199] for more information.

10.3. **Odds and Ends.** Deschamps and Ziolkowski have used Clifford algebra in their paper [59] to express the four Maxwell equations for electrodynamics in one relativistic four-dimensional spacetime equation. Similar to what they have done, we may use the orientation congruent algebra to also express the jump conditions as one relativistic four-dimensional spacetime equation:

$$\text{some equation.} \tag{10.6}$$

Just to try out a citation: [57].

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