Electromagnetism on an Exterior Derivative Flow Diagram with Plane Reflection Symmetry for Electromagnetic and Constitutive Duality[1]

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DRAFT & INCOMPLETE

Abstract. Flow diagrams for the differential equations of classical electromagnetism have previously been published as expository tools by Tonti and Deshamps. Electromagnetic duality, resulting from the theoretical magnetic monopole, and constitutive duality, such as between the **E** and **D** fields, are fundamental concepts of the theory. However, in neither the Tonti nor the Deshamps diagram do both dualities correspond to reflection in a line or plane. Hyperplane reflection, among all the nonidentical isometries (distance-preserving symmetry operations) of the Euclidean plane or space, is the only one that is both the most elementary from which the others can be composed, and also very simple, or even simplest, to visualize. We present a new flow diagram for the differential equations of the space-time split, (3+1)-dimensional, theory of classical electromagnetism in arbitrary media at rest. It is drawn as a lattice of rectangular parallelepipeds to allow two reflection planes. The horizontal reflection plane mirrors the field quantities with their electromagnetic duals; the vertical reflection plane mirrors the field quantities with their constitutive duals. As key to the elegant symbolic expression of these symmetries, we follow Burke by using both ordinary and twisted differential forms, possessing both inner and outer orientations, to write the equations in manifestly parity-invariant form. The previously unnamed "Lorentz functions" and "surge subsources" are represented. We exhibit Nisbet's direct plus dual gauge theory of the Hertz and stream potentials in this fitting arena. The origin of the diagram is briefly discussed.

Acknowledement

The author wishes to thank a close friend,
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1 Introduction: Building A Symmetrical Mansion for Electromagnetism

In general, flow diagrams provide a graphical representation of mathematical equations using nodes and arrows. A node represents a mathematical object; an arrow represents any operation that can be applied to the node object from which the arrow originates. An expression equal to a given node object is the sum of the operations of all arrows pointing to it, as those arrows operate on the node objects from which they originate.

The usefulness of flow diagrams in various presentations of the theory of classical electromagnetism has been demonstrated by Tonti, Deschamps, and many others.

Both the Tonti and Deschamps diagrams are arranged geometrically with certain arrow directions always associated with certain operations. This convention makes these two diagrams easy to interpret, although abstractly any helter-skelter arrangement of nodes and arrows with the same labels and incidences of arrows to or from nodes has exactly the same meaning. The geometric arrangement of the arrows in the Tonti and Deschamps diagrams also induces correspondences between geometric symmetries and physico-mathematical symmetries.

Two fundamental symmetries of the theory of classical electromagnetism are electromagnetic duality and constitutive duality. The term duality implies an involution — a transformation other than the identity that is its own inverse. Both of these dualities are so defined.

Electromagnetic duality results from the conceptual or theoretical existence of the magnetic monopole. On this basis theories of electromagnetism or electrodynamics, some of them controversial, have been constructed. Supplementing standard electromagnetic theory with the dual field quantities used as conceptual or computational aids is not controversial, although it is sometimes disparaged. Creating a new, mixed electrodynamic theory with the dual field quantities and particles interacting with the standard ones is very controversial.

Constitutive duality, such as between the **E** and **D** fields, occurs in the macroscopic theory as a description of the properties of material media, but also has a simple, linear counterpart in the microscopic theory involving only the constants ϵ_0 and μ_0, the permittivity and permeability of the vacuum.

How are these two dualities represented geometrically in the Tonti and Deshamps diagrams? We must answer that question in terms of the *extended* Tonti and Deshamps diagrams. By "extended" we mean the natural extension of their geometric arrangement to include all electromagnetic and constitutive dual quantities. Unfortunately, exactly how this is to be done is clear only after studying the new diagram and its relationship to the two existing ones. The extension, though, does not effect the 2- or 3-dimensional character of either diagram.

The extended Tonti diagram is 3-dimensional. Corresponding to electromagnetic duality it exhibits inversion symmetry in a central point; corresponding to constitutive duality it exhibits the symmetry of a shear followed by a rotation about a central vertical line followed by a shear opposite to the first one.

The extended Deshamps diagram is planar. Corresponding to electromagnetic duality it exhibits reflection symmetry in a central horizontal line; corresponding to constitutive duality it exhibits a permutation symmetry that might be considered as a "local" reflection of the nodes immediately above and below four different horizontal lines.

In neither the Tonti nor the Deshamps diagram do both dualities correspond to reflection in a line or plane — an isometry (distance-preserving transformtion) which is also a geometric involution, just as the two dualities are physico-mathematical involutions. In addition, hyperplane reflection, among all the nonidentical isometries of the Euclidean plane or space, is the only one that is both the most elementary from which the others can be composed, and also very simple, or even simplest, to visualize.

We present a new flow diagram representing the differential equations of the space-time split, (3+1)-dimensional, theory of classical electromagnetism in arbitrary media at rest. It is drawn as a lattice of rectangular parallelepipeds to allow two reflection planes. The horizontal reflection plane mirrors the field quantities with their electromagnetic duals; the vertical reflection plane mirrors the field quantities with their constitutive duals.

The elegance of this geometric symmetry will be equalled symbolically by following Burke in using the formalism of ordinary (inner-oriented) and twisted (outer-oriented) differential forms to write the equations of classical electomagnetism in manifestly parity-invariant form.

For example, the 3-form of electric charge density is twisted (has an outer orientation); thus, in manifestly parity-invariant form, it carries a positive or negative sign. Its electomagnetic dual, the 3-form of magnetic charge density is ordinary (has an inner orientation); thus, in manifestly parity-invariant form, it carries a right- or left-handed screwsense. Further, the constitutive duals are constructed with the *twisted* Hodge star operator.

The new flow diagram, in addition to literally mirroring the two dualities graphically, is more suitable for representing equations involving the Laplace-Beltrami Δ and d'Alembertian $\Box$ operators. Although these two operators are not explicitly shown in the new diagram, their component operators, the exterior derivative d and the coderivative δ are shown, permitting Δ and $\Box$ to have a symmetrical composite graphical representation. This symmetry is impossible in the extended Tonti diagram.

Among the quantities placed in the new flow diagram are the less familiar Hertz "superpotentials' and previously unnamed "surge subsources," and "Lorentz functions." Altogher the new diagram represents forty-eight physical quantities with all electromagnetic and constitutive dual pairs included.

The Tonti and Deshamps diagrams are adequate for the purposes of their creators. But by symmetrically including the coderivative operator, and all electromagnetic and constitutive duals of the standard field quantities the new flow diagram may be regared as the symmetrical completion of the Tonti and Deshamps diagrams.

One may ask if a cycle of transformations exists taking the extended Tonti and Deschamps diagrams and the new one into each other. Such a cycle does exist; however, none of its transformations are isometries. The exposition of this cycle will have to wait for a possible future publication.

While some may regard this new diagram (or the earlier two) as trivial because they merely hang the same differential equations of electromagnetism upon various graphical skeletons, the author believes that physico-mathematical relationships among the field quantities of classical electromagnetism are more easily grasped when depicted by corresponding graphical relationships.

In the sequel we use the new flow diagram to graphically illustrate

i) the potentials, Maxwell's equations, and the equation of continuity (Figure 1)

ii) the electromagnetic dual of item i) (Figure 2)

iii) the E and B wave equations for electric sources (Figure 3)

iv) the potential wave equations for electric sources (Figure 4)

v) Nisbet's Hertz and stream potential equations *sans* gauge terms (Figure 5)

vi) Nisbet's direct and dual guage terms for item v) (Figure 6).

2 Details of the New Home as Maxwell and Immediate Family Move In

2.1 Graphical and Symbolic Conventions

This diagram is similar to a room occupancy chart. Several guests may be staying in the same room (node). Also some of the rooms may have sliding partitions to subdivide them. The rooms in this mansion are not numbered but labelled with the names of principal occupants. The room labels are engraved on little plaques such as $\lfloor E \rfloor$ and $\lfloor B \rfloor$. Of course this does not mean that Ms. E is always occupying room $\lfloor E \rfloor$, nor is she always the sole occupant, but any resident of her room must be a very close relative of hers. In other words the labels are conventional and do not necessarily indicate the actual contents of a node. The relationship between a node label and its contents is idicated by symbolic and graphical conventions.

Those conventions are ...

Arrows lying along the main diagonals of the rectangular parallelepipeds ("diagonal arrows" for short) represent the 3-space derivative operators, the exterior derivative d, and the coderivative δ. These diagonal arrows may carry a negative sign. If a diagonal arrow is solid it represents the exterior derivative d; if it is dashed it represents the coderivative δ. Horizontal arrows directed rightward represent the time partial derivative operator ∂_t. These horizontal arrows may carry a negative sign or a multiplier which is the inverse square of the vacuum light speed c^{-2}. A dashed right-going horizontal arrow indicates the presence of the multiplier c^{-2}; nondashed right-going horizontal arrows do not carry this multiplier. Horizontal arrows directed forwards and backwards represent the constitutive duality operator. These horizontal arrows carry no sign or multiplier. Their fully or partially open arrowheads indicate that they are not to be added to the sum equal to a node and formed by the operation of the other arrows pointing to it. Instead they indicate an independent equality of the terminating node with result of the indicated constitutive duality operator acting on the originating node. (In general only arrows with fully filled heads contribute to a node sum.) Those with fully open heads indicate a constitutive duality operation involving the permittivity operator $\epsilon\tilde{*}$; Those with half open heads indicate a constitutive duality operation involving the permeability operator $\mu\tilde{*}^{-1}$. Following Deschamps a small bar perpendicular to any arrow (and usually placed as close to its tail as possible) indicates the operator is preceded by a negative sign.

We use several conventions to save space inside the cramped nodes. One of these is the indication of algebraic inversion by typographical inversion. The unit of resistance, the Ohm, provides a familiar example of this typographical convention. The symbol for the Ohm is Ω; a common symbol for its inverse is ℧. (This symbol, though, is not standard in the SI; in the International System of Units the the unit of conductance is the siemens, symbolized by S.)

We use Burke's notation for the twisted Hodge star operator $\tilde{*}$. The full notations for its forms involving premultiplication by the free space permittivity and permeability are $\epsilon_0\tilde{*}$ and $\mu_0\tilde{*}^{-1}$. For economy of space, however, these will frequently be written as $\tilde{\epsilon}_0$ and $\utilde{\mu}_0$. The inverses of these operators given in full notation are $\epsilon_0^{-1}\tilde{*}^{-1}$ and $\mu_0^{-1}\tilde{*}$; with the above typographical inversion convention they become $\utilde{\rotatebox[origin=c]{180}{ϵ}}_0$ and $\tilde{\rotatebox[origin=c]{180}{μ}}_0$.

We write the inverse square of the vacuum light speed $c^{-2} = \tilde{\epsilon}_0\utilde{\mu}_0 = \epsilon_0\mu_0$ compactly as the arbitrary symbol s.

Twelve nodes of the diagram are "mixed" in that they may contain nonzero, nongauge terms of both electrical and magnetic origin. Four familiar ones form a rectangle at the heart of the diagram. These are $\lfloor E\rfloor$, $\lfloor\widetilde{D}\rfloor$, $\lfloor\widetilde{H}\rfloor$, and $\lfloor B\rfloor$. The remaining eight less familiar nodes lie on similar rectangles displaced to the far left and right. We have labelled the nodes of the left rectangle $\lfloor\Pi\rfloor$, $\lfloor\tilde{\epsilon}\,\Pi\rfloor$, $\lfloor\tilde{\rotatebox[origin=c]{180}{μ}}\Sigma\rfloor$, and $\lfloor\Sigma\rfloor$, and the nodes of the right rectangle $\lfloor\utilde{\rotatebox[origin=c]{180}{ϵ}}\tilde{\zeta}\rfloor$, $\lfloor\tilde{\zeta}\rfloor$, $\lfloor\tilde{\xi}\rfloor$, and $\lfloor\utilde{\mu}\tilde{\xi}\rfloor$. This labelling breaks the EM duality symmetry of the diagram and shows our bias toward the actual existence of electric charge rather than the hypothetical existence of magnetic charge. One possible symmetrical labelling, though, can be produced by replacing two labels of the left rectangle as $\lfloor\tilde{\rotatebox[origin=c]{180}{μ}}\Sigma\rfloor\to\lfloor\Sigma\rfloor$ and $\lfloor\Sigma\rfloor\to\lfloor\utilde{\mu}\Sigma\rfloor$, and two of the right rectangle as $\lfloor\tilde{\xi}\rfloor\to\lfloor\tilde{\rotatebox[origin=c]{180}{μ}}\xi\rfloor$ and $\lfloor\utilde{\mu}\tilde{\xi}\rfloor\to\lfloor\xi\rfloor$. A second way to acheive symmetry for these eight nodes is by labelling them with single symbols such as we have already done for the core nodes $\lfloor E\rfloor$, $\lfloor\widetilde{D}\rfloor$, $\lfloor\widetilde{H}\rfloor$, and $\lfloor B\rfloor$.

2.2 Units

To emphasize the differences between ordinary and twisted forms we use the SI system of units. Constitutive duality operations taking an ordinary differential form to a twisted one will always involve a multiplication by $\epsilon_0$or $\eta_0$in SI units.

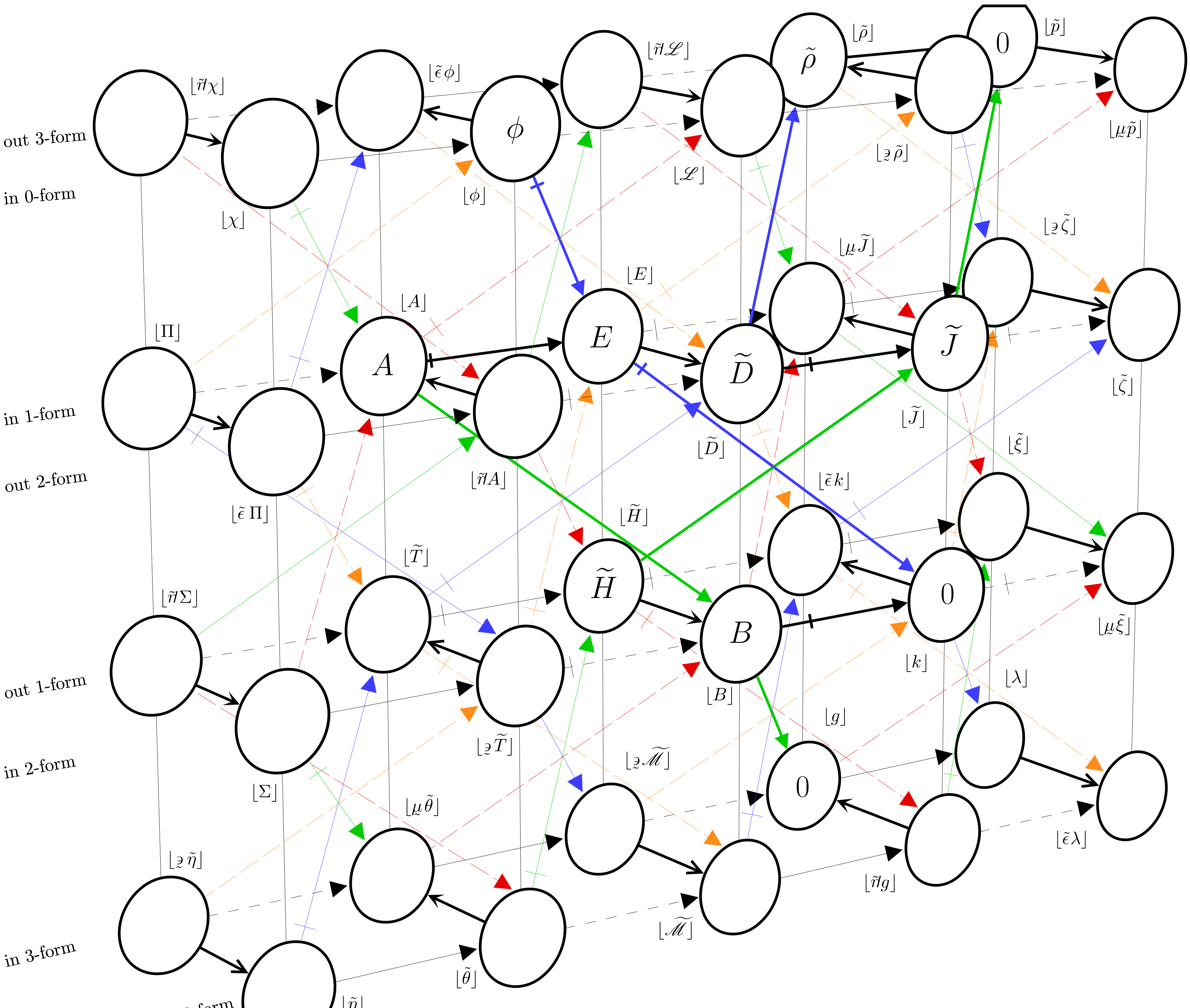


Figure 1: The Potentials, Maxwell's Equations, and the Continuity Equation for Electric Sources

3 Maxwell's Southern Cousin and Friends Visit: The Electromagnetic Duals of the Fundamental Equations

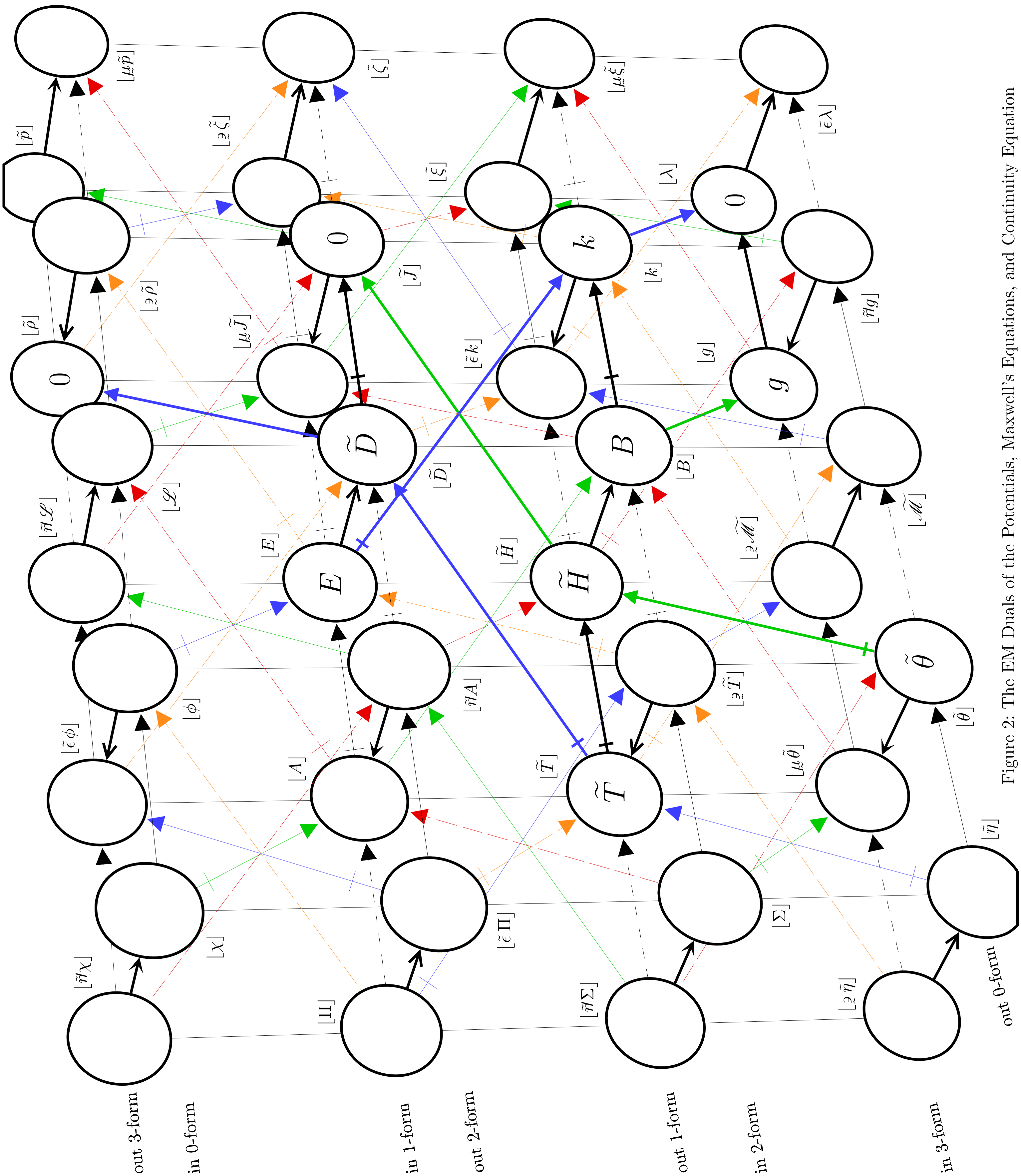


Figure 2: The EM Duals of the Potentials, Maxwell's Equations, and Continuity Equation

4 Maxwell's Eastern Cousins Pay a Visit: The E and B Wave Equations

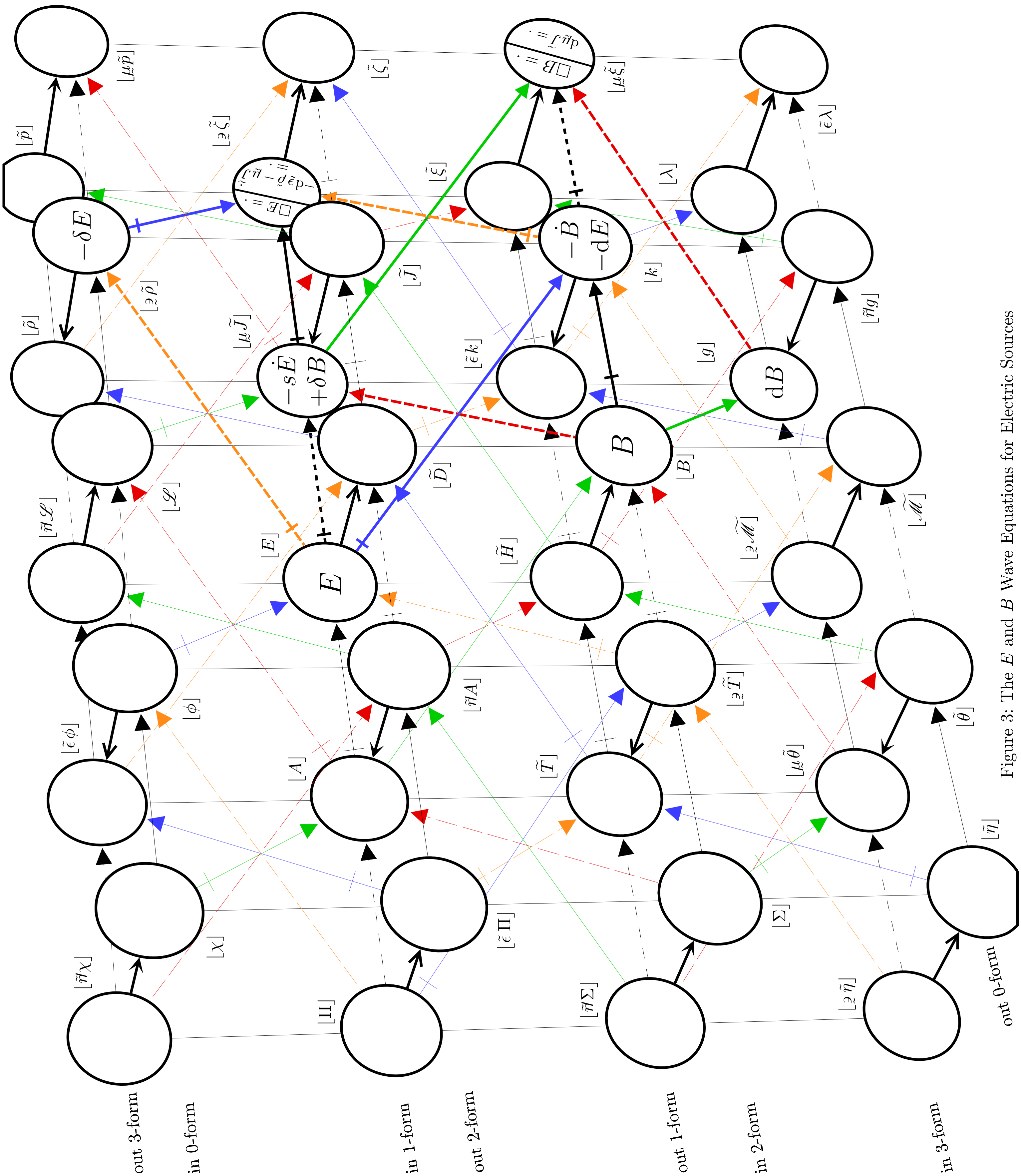


Figure 3: The E and B Wave Equations for Electric Sources

5 Some Other Passageways Among the Family's Rooms: The Potential Wave Equations

$$\mathrm{d}\widetilde{D} = \tilde{\rho} \tag{5.1a}$$
$$\mathrm{d}(\epsilon\tilde{*}E) = \tilde{\rho} \tag{5.1b}$$
$$\epsilon^{-1}\tilde{*}^{-1}\,\mathrm{d}(\epsilon\tilde{*}E) = \epsilon^{-1}\tilde{*}^{-1}\,\tilde{\rho} \tag{5.1c}$$
$$-\delta E = \epsilon^{-1}\tilde{*}^{-1}\,\tilde{\rho} \tag{5.1d}$$
$$-\delta(-\mathrm{d}\phi - \partial_t A) = \epsilon^{-1}\tilde{*}^{-1}\,\tilde{\rho} \tag{5.1e}$$
$$\Delta\phi + \partial_t \delta A = \epsilon^{-1}\tilde{*}^{-1}\,\tilde{\rho} \tag{5.1f}$$
$$\Delta + c^{-2}\partial_t{}^2\phi = \epsilon^{-1}\tilde{*}^{-1}\,\tilde{\rho} \tag{5.1g}$$
$$\Box\phi = \epsilon^{-1}\tilde{*}^{-1}\,\tilde{\rho} \tag{5.1h}$$

$$\mathrm{d}\widetilde{H} - \partial_t\widetilde{D} = \widetilde{J} \tag{5.2a}$$
$$\mathrm{d}(\mu^{-1}\tilde{*}B) - \partial_t(\epsilon\tilde{*}E) = \widetilde{J} \tag{5.2b}$$
$$\mu\tilde{*}^{-1}\mathrm{d}(\mu^{-1}\tilde{*}B) - \mu\tilde{*}^{-1}\partial_t(\epsilon\tilde{*}E) = \mu\tilde{*}^{-1}\widetilde{J} \tag{5.2c}$$
$$\delta B - c^{-2}\partial_t E = \mu\tilde{*}^{-1}\widetilde{J} \tag{5.2d}$$
$$\delta\mathrm{d}A - c^{-2}\partial_t(-\mathrm{d}\phi - \partial_t A) = \mu\tilde{*}^{-1}\widetilde{J} \tag{5.2e}$$
$$\delta\mathrm{d}A + c^{-2}\mathrm{d}\partial_t\phi + c^{-2}\partial_t{}^2 A = \mu\tilde{*}^{-1}\widetilde{J} \tag{5.2f}$$
$$\delta\mathrm{d} + \mathrm{d}\delta + c^{-2}\partial_t{}^2 A = \mu\tilde{*}^{-1}\widetilde{J} \tag{5.2g}$$
$$\Delta + c^{-2}\partial_t{}^2 A = \mu\tilde{*}^{-1}\widetilde{J} \tag{5.2h}$$
$$\Box A = \mu\tilde{*}^{-1}\widetilde{J} \tag{5.2i}$$

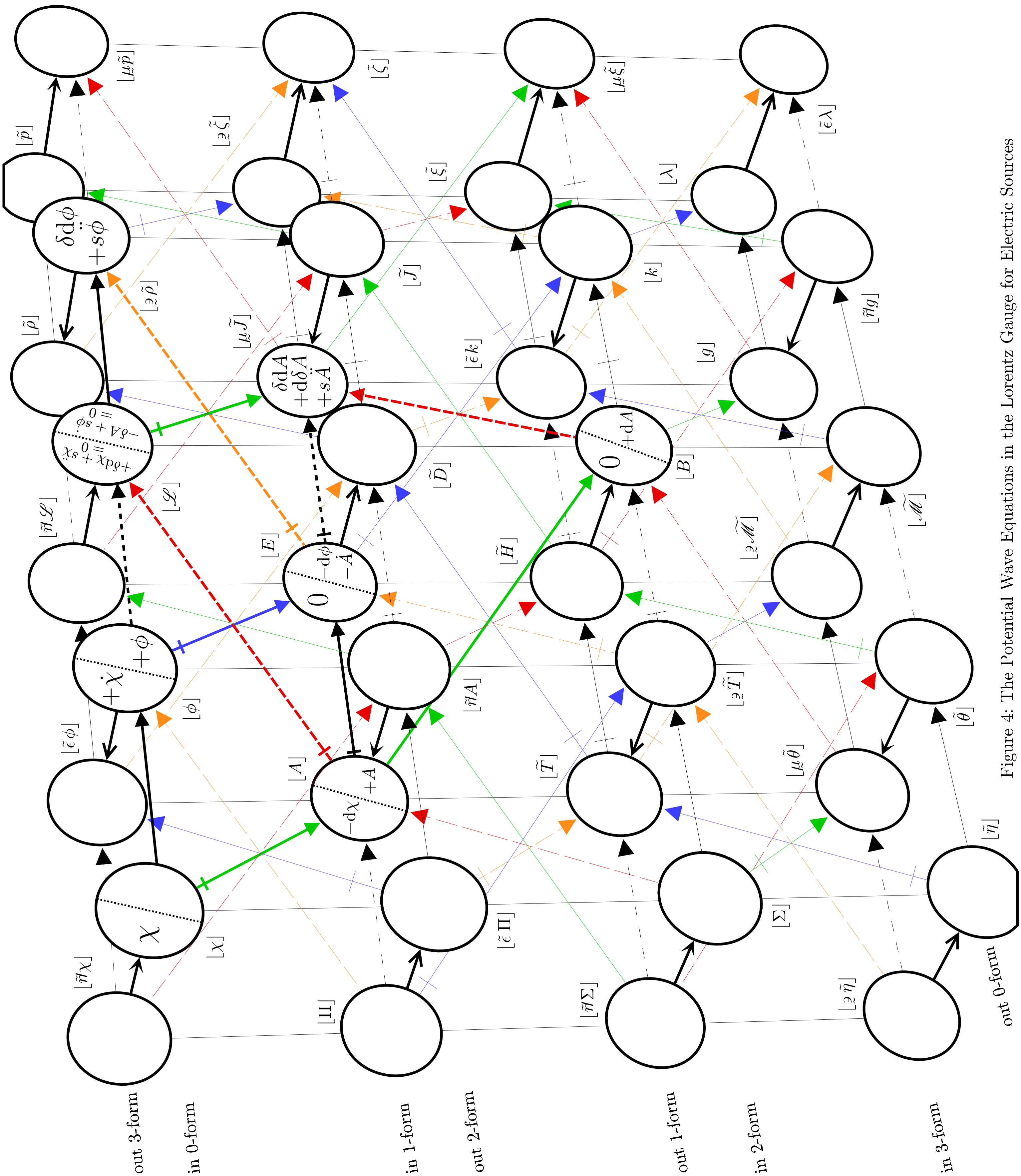


Figure 4: The Potential Wave Equations in the Lorentz Gauge for Electric Sources

6 Maxwell's Cousins from out West Come Calling: The Hertz "Superpotentials" and Their Guages

6.1 Nisbet's Equations Written with Ordinary and Twisted Differential Forms

$$-\mathrm{d}E - \dot{B} = {}^{e}k = 0, \qquad \mathrm{d}B = {}^{e}g = 0 \tag{N 2·1}$$

$$\mathrm{d}\widetilde{H} - \dot{\widetilde{D}} = \widetilde{J}, \qquad \mathrm{d}\widetilde{D} = \tilde{\rho} \tag{N 2·2}$$

$$\widetilde{D} = \tilde{\epsilon}_0 E + \widetilde{P}, \qquad \widetilde{H} = \tilde{\reflectbox{μ}}_0 B - \widetilde{M} \tag{N 2·3}$$

$$E = -\mathrm{d}\phi - \dot{A}, \qquad B = \mathrm{d}A \tag{N 2·4}$$

$$-\delta A + c^{-2}\dot{\phi} = \mathscr{L} = 0 \tag{N 2·5}$$

$$\Box\phi = \utilde{\ni}_0 \tilde{\rho} + \delta \utilde{\ni}_0 \widetilde{P} \tag{N 2·6}$$

$$\Box A = \utilde{\mu}_0 \widetilde{J} + \delta \utilde{\mu}_0 \widetilde{M} + \utilde{\mu}_0 \dot{\widetilde{P}} \tag{N 2·7}$$

$$\begin{gathered} \Box \triangleq \Delta + c^{-2}{\partial_t}^2 \triangleq \delta\mathrm{d} + \mathrm{d}\delta + c^{-2}{\partial_t}^2, \text{ where} \\ c^{-2} \triangleq c^{-2} = \ni_0 \reflectbox{μ}_0 = \epsilon_0^{-1}\mu_0^{-1} \end{gathered} \tag{N 2·8}$$

$$\phi = \delta\Pi, \qquad A = \delta\Sigma + c^{-2}\dot{\Pi} \tag{N 2·9}$$

$$E = -\mathrm{d}\delta\Pi - c^{-2}\ddot{\Pi} - \delta\dot{\Sigma} \tag{N 2·10}$$

$$B = \mathrm{d}\delta\Sigma + c^{-2}\mathrm{d}\dot{\Pi} \tag{N 2·11}$$

$$\tilde{\rho} = -\mathrm{d}\widetilde{Q}_e, \qquad \widetilde{J} = \mathrm{d}\widetilde{Q}_m + \dot{\widetilde{Q}}_e \tag{N 2·12}$$

$$\mathrm{d}\widetilde{J} + \dot{\tilde{\rho}} = \tilde{p} = 0 \tag{N 2·13}$$

$$\Box\Pi = \utilde{\ni}_0 \widetilde{P} + \utilde{\ni}_0 \widetilde{Q}_e \tag{N 2·14}$$

$$\Box\Sigma = \utilde{\mu}_0 \widetilde{M} + \utilde{\mu}_0 \widetilde{Q}_m \tag{N 2·15}$$

$$E = R_e - \mathrm{d}\delta\Pi - c^{-2}\ddot{\Pi} - \delta\dot{\Sigma} \tag{N 2·18}$$

$$B = -R_m + \mathrm{d}\delta\Sigma + c^{-2}\mathrm{d}\dot{\Pi} \tag{N 2·19}$$

$$\widetilde{D} = -\widetilde{Q}_e + \mathrm{d}\delta\tilde{\epsilon}_0\Pi - \mathrm{d}\tilde{\epsilon}_0\dot{\Sigma} \tag{N 2·20}$$

$$\widetilde{H} = \widetilde{Q}_m - \mathrm{d}\delta\,\tilde{\eta}_0\Sigma - c^{-2}\,\tilde{\eta}_0\ddot{\Sigma} + c^{-2}\delta\,\tilde{\eta}_0\dot{\Pi} \tag{N 2·21}$$

$$\Box\Pi = \underset{\sim}{\epsilon}_0\widetilde{P} + \underset{\sim}{\epsilon}_0\widetilde{Q}_e + R_e \tag{N 2·22}$$

$$\Box\Sigma = \underset{\sim}{\mu}_0\widetilde{M} + \underset{\sim}{\mu}_0\widetilde{Q}_m + R_m \tag{N 2·23}$$

$$g = -\mathrm{d}R_m, \qquad k = -\mathrm{d}R_e + \dot{R}_m \tag{N 2·24}$$

$$\phi = \phi^0 + \dot{\chi}, \qquad A = A^0 - \mathrm{d}\chi \tag{N 3·1}$$

$$\Box\chi = 0 \tag{N 3·2}$$

$$\left.\begin{aligned} \widetilde{Q}_e &= \widetilde{Q}_e^0 + \mathrm{d}\widetilde{G}, \\ \widetilde{Q}_m &= \widetilde{Q}_m^0 - \mathrm{d}\tilde{g} - \dot{\widetilde{G}} \end{aligned}\right\} \tag{N 3·3}$$

$$\left.\begin{aligned} R_e &= R_e^0 - \mathrm{d}l - \dot{L}, \\ R_m &= R_m^0 - \mathrm{d}L \end{aligned}\right\} \tag{N 3·4}$$

$$E = R_e^0 - \mathrm{d}l - \dot{L} - \mathrm{d}\delta\Pi - c^{-2}\ddot{\Pi} - \delta\dot{\Sigma} \tag{N 3·6}$$

$$B = -R_m^0 + \mathrm{d}L + \mathrm{d}\delta\Sigma + c^{-2}\mathrm{d}\dot{\Pi} \tag{N 3·7}$$

$$\widetilde{D} = -\widetilde{Q}_e^0 - \mathrm{d}\widetilde{G} + \mathrm{d}\delta\tilde{\epsilon}_0\Pi - \mathrm{d}\tilde{\epsilon}_0\dot{\Sigma} \tag{N 3·8}$$

$$\widetilde{H} = \widetilde{Q}_m^0 - \mathrm{d}\tilde{\not{g}} - \dot{\widetilde{G}} - \mathrm{d}\delta\,\tilde{\not\eta}_0\Sigma - c^{-2}\,\tilde{\not\eta}_0\ddot{\Sigma} + c^{-2}\delta\,\tilde{\not\eta}_0\dot{\Pi} \tag{N 3·9}$$

$$\Box\Pi = \underset{\sim}{\epsilon}_0\widetilde{P} + \underset{\sim}{\epsilon}_0\widetilde{Q}_e^0 + R_e^0 + \mathrm{d}\underset{\sim}{\epsilon}_0\widetilde{G} - \mathrm{d}l - \dot{L} \tag{N 3·10}$$

$$\Box\Sigma = \underset{\sim}{\mu}_0\widetilde{M} + \underset{\sim}{\mu}_0\widetilde{Q}_m^0 + R_m^0 + \delta\underset{\sim}{\mu}_0\tilde{\not{g}} - \underset{\sim}{\mu}_0\dot{\widetilde{G}} - \mathrm{d}L \tag{N 3·11}$$

$$\left.\begin{aligned} \Pi &= \Pi^0 - \mathrm{d}\not\lambda + \delta\Gamma - \dot{\Lambda}, \\ \Sigma &= \Sigma^0 + \delta\gamma - \mathrm{d}\Lambda - c^{-2}\dot{\Gamma} \end{aligned}\right\} \tag{N 3·12}$$

$$\Box\not\lambda = -\dot{\not\zeta}, \qquad \Box\Lambda = \mathrm{d}\not\zeta \tag{N 3·13}$$

$$\chi = \not\zeta - \delta\Lambda + c^{-2}\dot{\not\lambda} \tag{N 3·14}$$

$$\tilde{\not{g}} = \Box\,\tilde{\not\eta}_0\gamma + \dot{\tilde{\not\xi}}, \qquad \widetilde{G} = \Box\tilde{\epsilon}_0\Gamma - \mathrm{d}\tilde{\not\xi} \tag{N 3·15}$$

$$l = \Box\not\lambda + \dot{\not\zeta}, \qquad L = \Box\Lambda - \mathrm{d}\not\zeta \tag{N 3·16}$$

6.2 Derivations of Hertz and Stream Potential Wave Equations

$$-\delta E = -\delta E \tag{6.1a}$$

$$-\delta(-\mathrm{d}\delta\Pi - c^{-2}\ddot{\Pi} - \delta\dot{\Sigma}) = -\delta(\utilde{\epsilon}_0\widetilde{D} - \utilde{\epsilon}_0\widetilde{P}) \tag{6.1b}$$

$$\delta\mathrm{d}\delta\Pi + c^{-2}\delta\ddot{\Pi} = -\delta(-\utilde{\epsilon}_0\widetilde{Q}_e - \utilde{\epsilon}_0\widetilde{P}) \tag{6.1c}$$

$$\delta\Box\Pi = \delta(\utilde{\epsilon}_0\widetilde{Q}_e + \utilde{\epsilon}_0\widetilde{P}) \tag{6.1d}$$

$$\Box\Pi^0 = \utilde{\epsilon}_0\widetilde{Q}_e^0 + \utilde{\epsilon}_0\widetilde{P} \tag{6.1e}$$

$$\delta B - c^{-2}\dot{E} = \delta B - c^{-2}\dot{E} \tag{6.2a}$$

$$\delta(\mathrm{d}\delta\Sigma + c^{-2}\mathrm{d}\dot{\Pi}) - c^{-2}\partial_t(-\mathrm{d}\delta\Pi - c^{-2}\ddot{\Pi} - \delta\dot{\Sigma}) = \delta(\utilde{\mu}_0\widetilde{H} + \utilde{\mu}_0\widetilde{M}) - c^{-2}\partial_t(\utilde{\epsilon}_0\widetilde{D} - \utilde{\epsilon}_0\widetilde{P}) \tag{6.2b}$$

$$(\delta\mathrm{d}\delta\Sigma + c^{-2}\delta\ddot{\Sigma}) + c^{-2}\partial_t(\delta\mathrm{d}\Pi + \mathrm{d}\delta\Pi + c^{-2}\ddot{\Pi}) = \delta(\utilde{\mu}_0\widetilde{Q}_m + \utilde{\mu}_0\widetilde{M}) - c^{-2}\partial_t(-\utilde{\epsilon}_0\widetilde{Q}_e - \utilde{\epsilon}_0\widetilde{P}) \tag{6.2c}$$

$$\delta\Box\Sigma + c^{-2}\partial_t\Box\Pi = \delta(\utilde{\mu}_0\widetilde{Q}_m + \utilde{\mu}_0\widetilde{M}) + c^{-2}\partial_t(+\utilde{\epsilon}_0\widetilde{Q}_e + \utilde{\epsilon}_0\widetilde{P}) \tag{6.2d}$$

$$\Box\Sigma^0 = \utilde{\mu}_0\widetilde{Q}_m^0 + \utilde{\mu}_0\widetilde{M} \tag{6.2e}$$

6.3 The Grand Four Panel Diagrams of Nisbet's Gauge Theory of the Hertz and Stream Potentials

Figure 5(a). Derivation of the Hertz and Stream Potential Equations without Gauge Terms (top left panel of a four page composite)

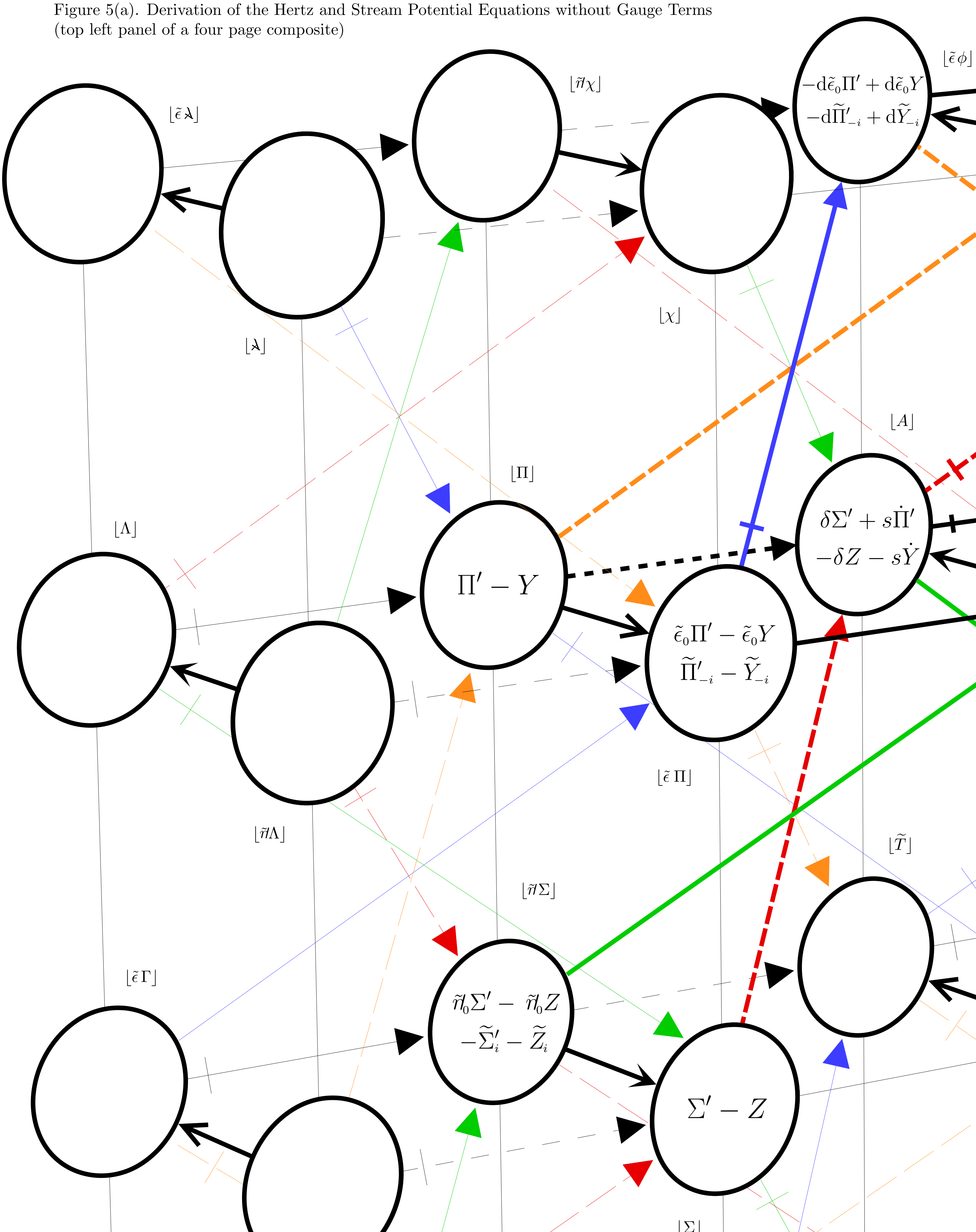
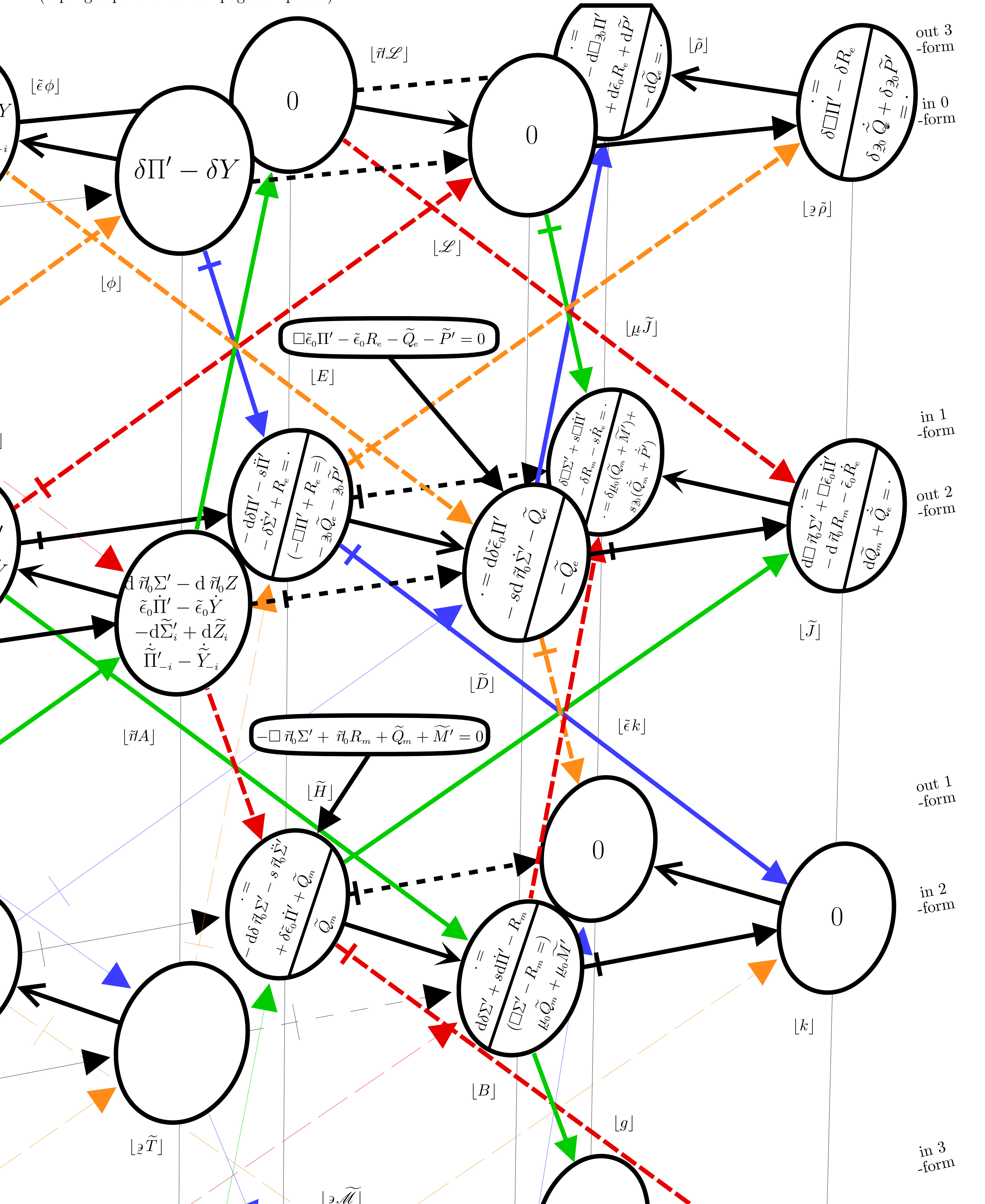


Figure 5(b). Deriv. of Hertz & Stream Pot. Eqs. w/o Gauges
(top right panel of a four page composite)



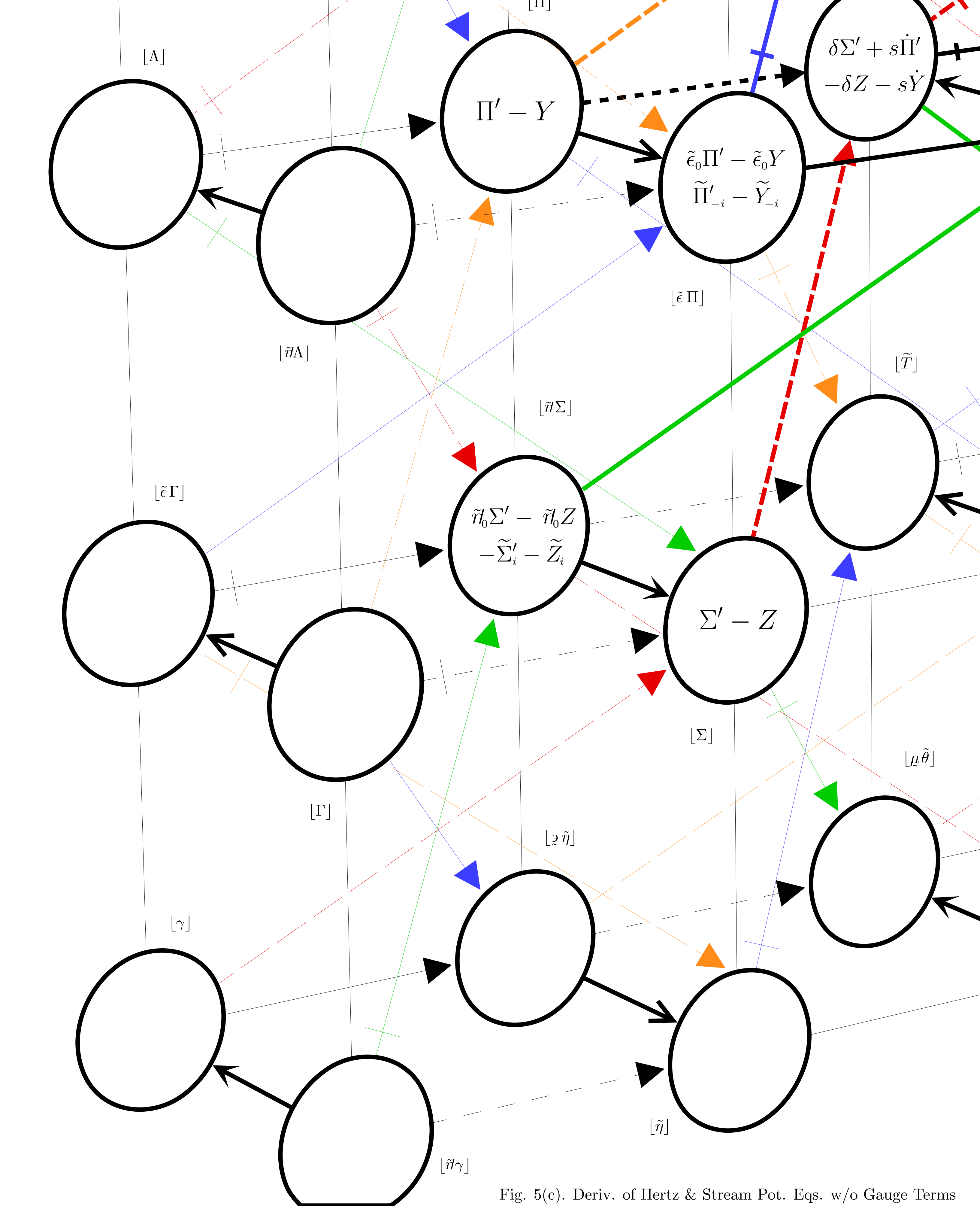


Fig. 5(c). Deriv. of Hertz & Stream Pot. Eqs. w/o Gauge Terms
(bottom left panel of a four page composite)

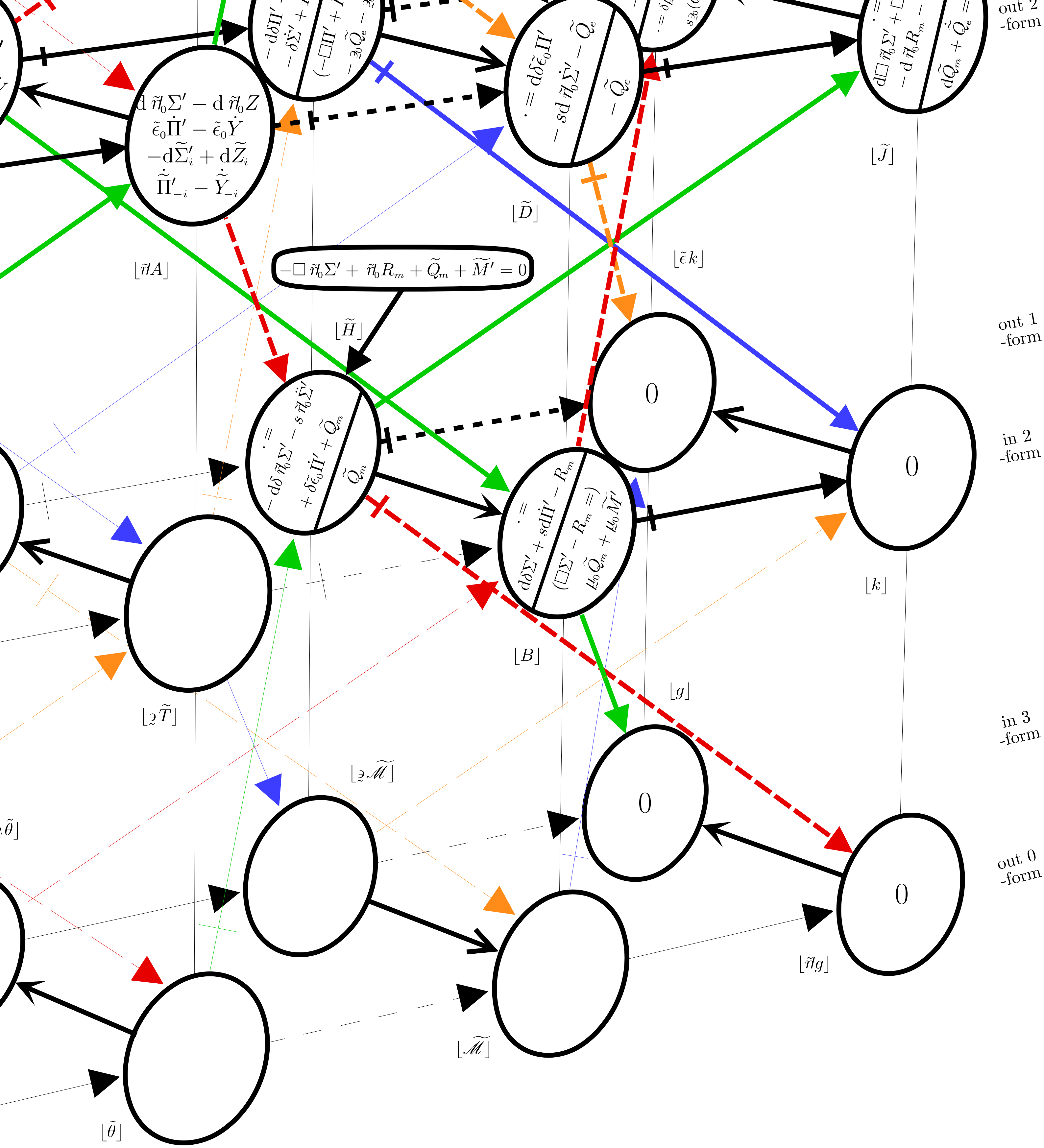


Figure 5(d). Derivation of the Hertz and Stream Potential Equations without Gauge Terms
(bottom right panel of a four page composite)

Figure 6(a). Derivation of the Gauge Equations for the Hertz and Stream Potentials
(top left panel of a four page composite)

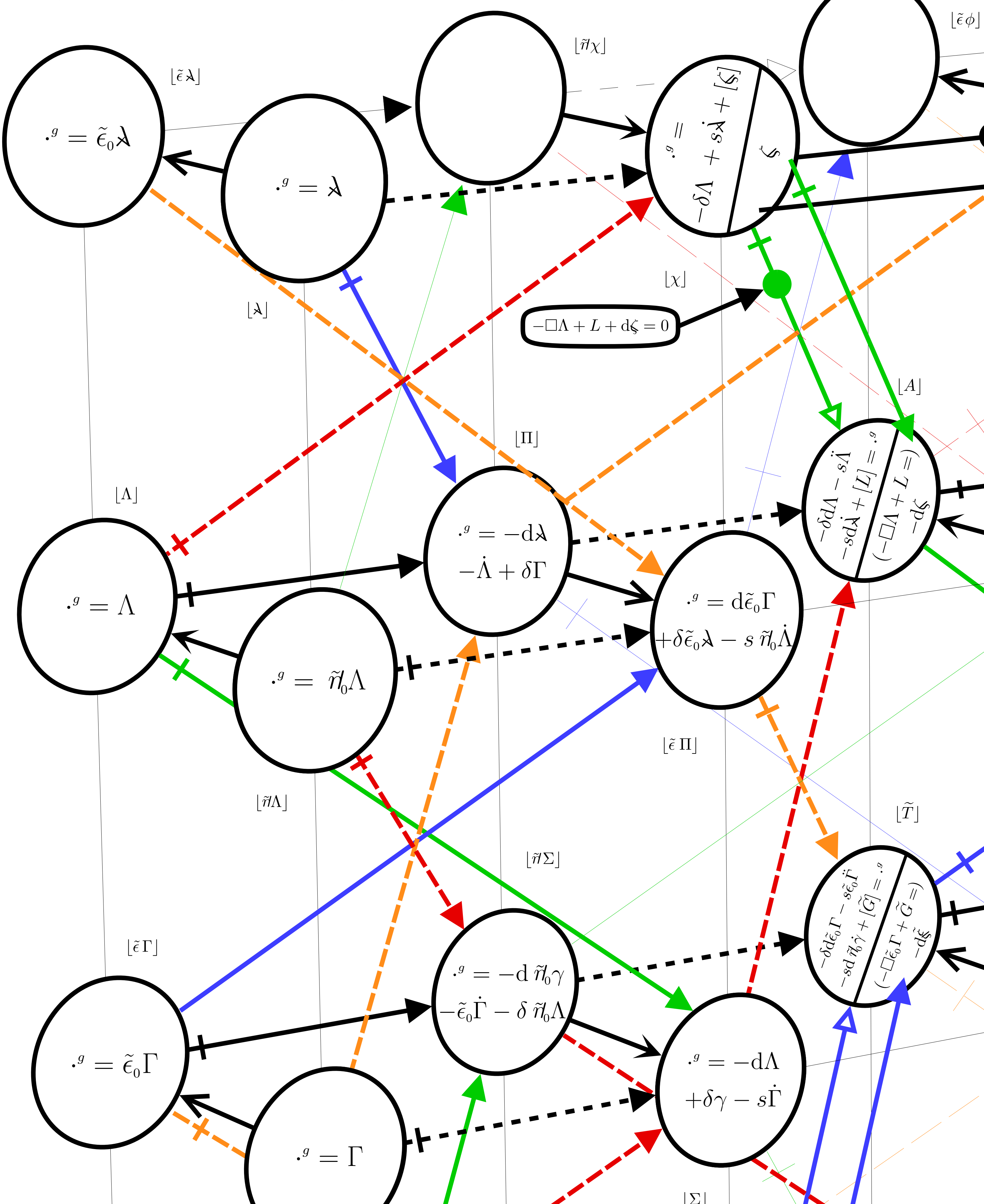
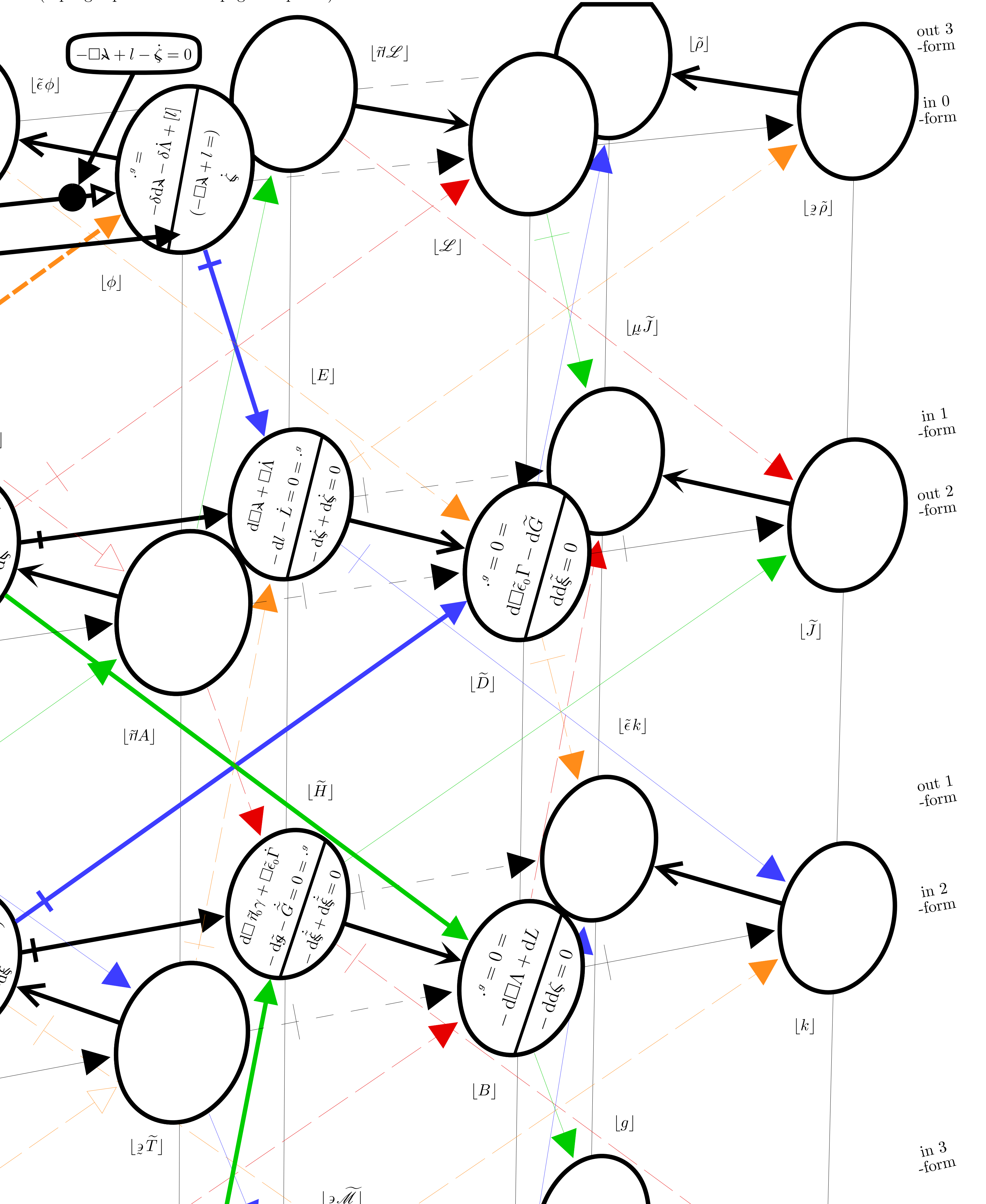


Figure 6(b). Deriv. of Gauge Eqs. for Hertz & Stream Pots.
(top right panel of a four page composite)



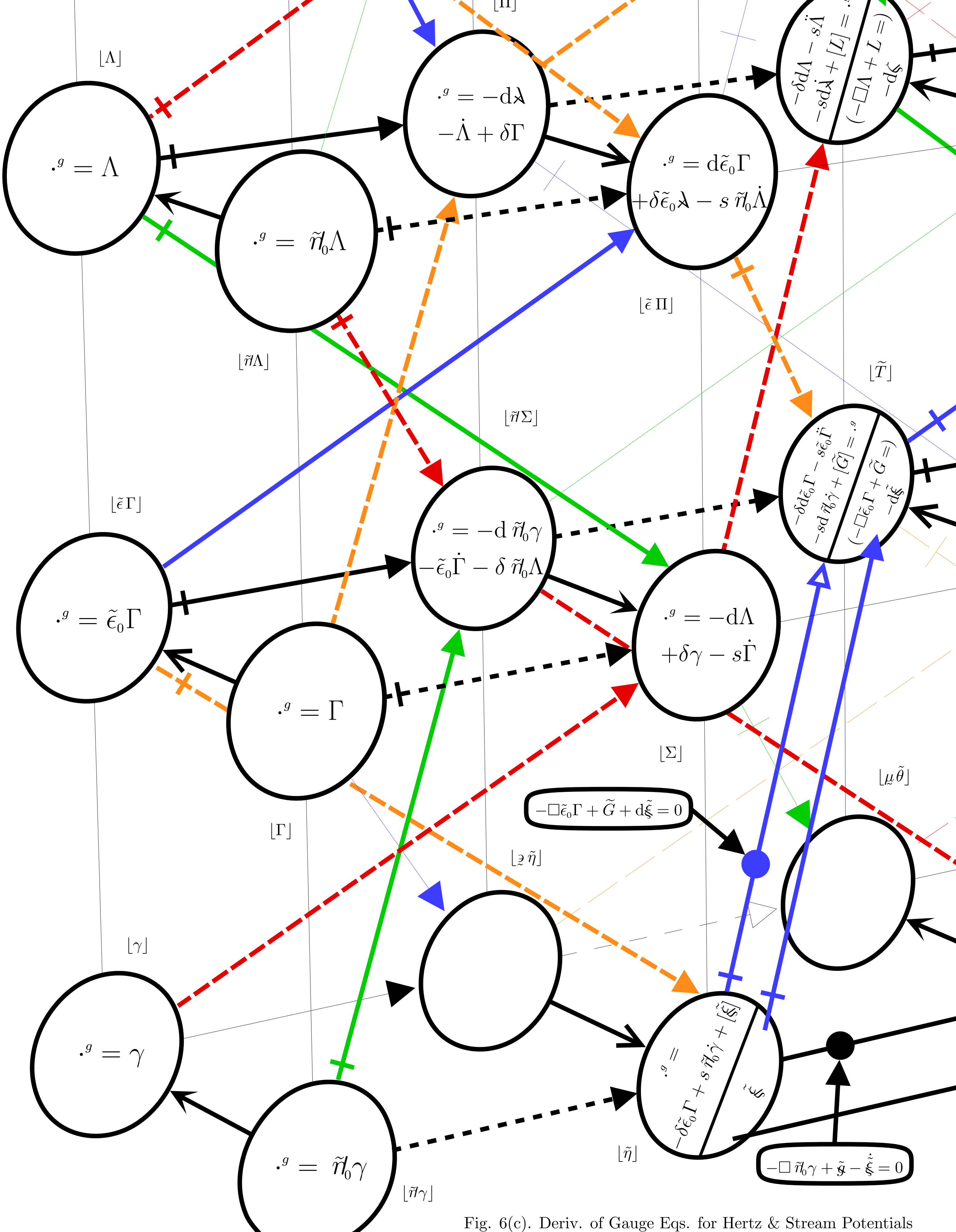


Fig. 6(c). Deriv. of Gauge Eqs. for Hertz & Stream Potentials
(bottom left panel of a four page composite)

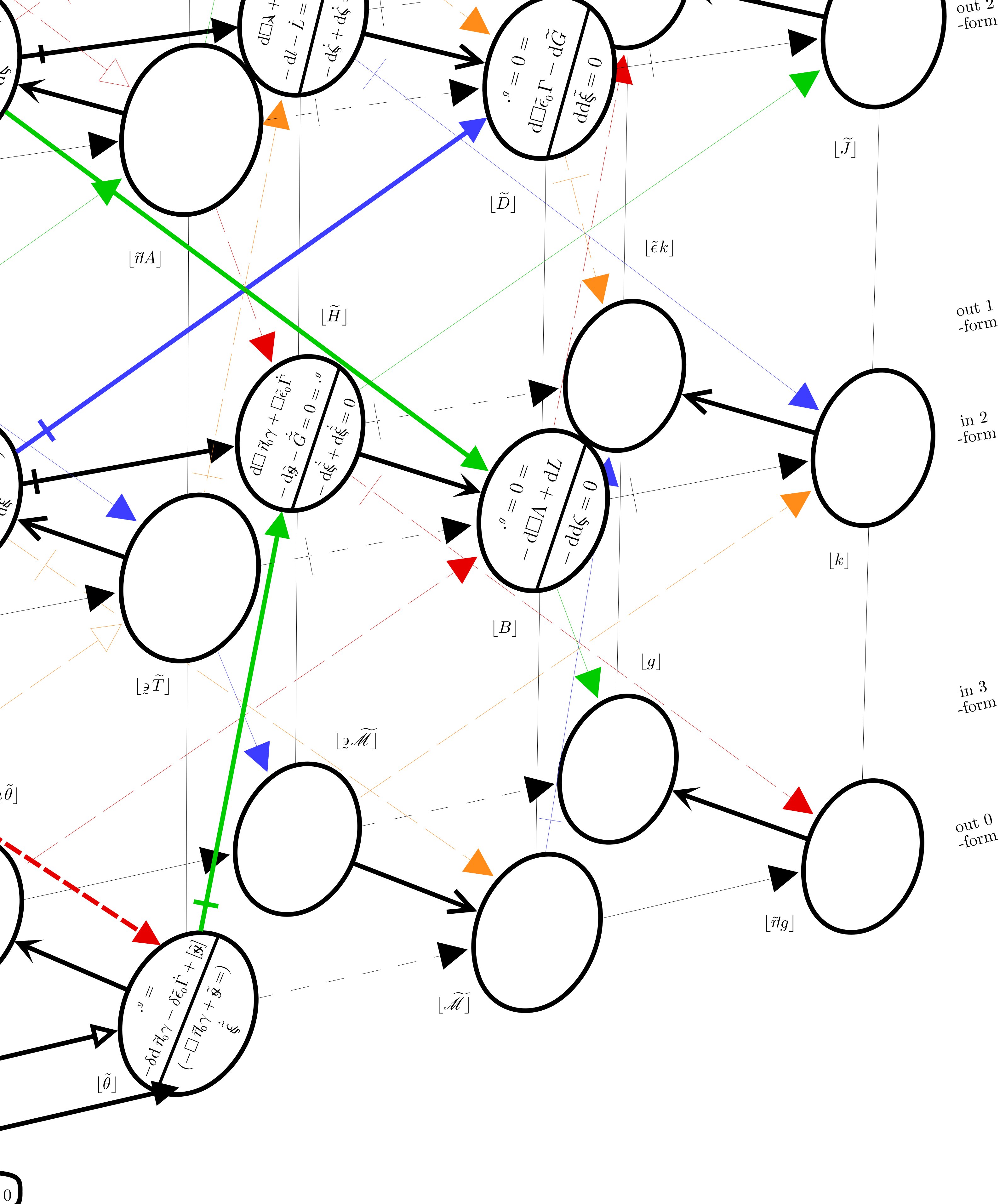


Figure 6(d). Derivation of the Gauge Equations for the Hertz and Stream Potentials
(bottom right panel of a four page composite)

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7 A Blank Diagram for the Reader to Play With

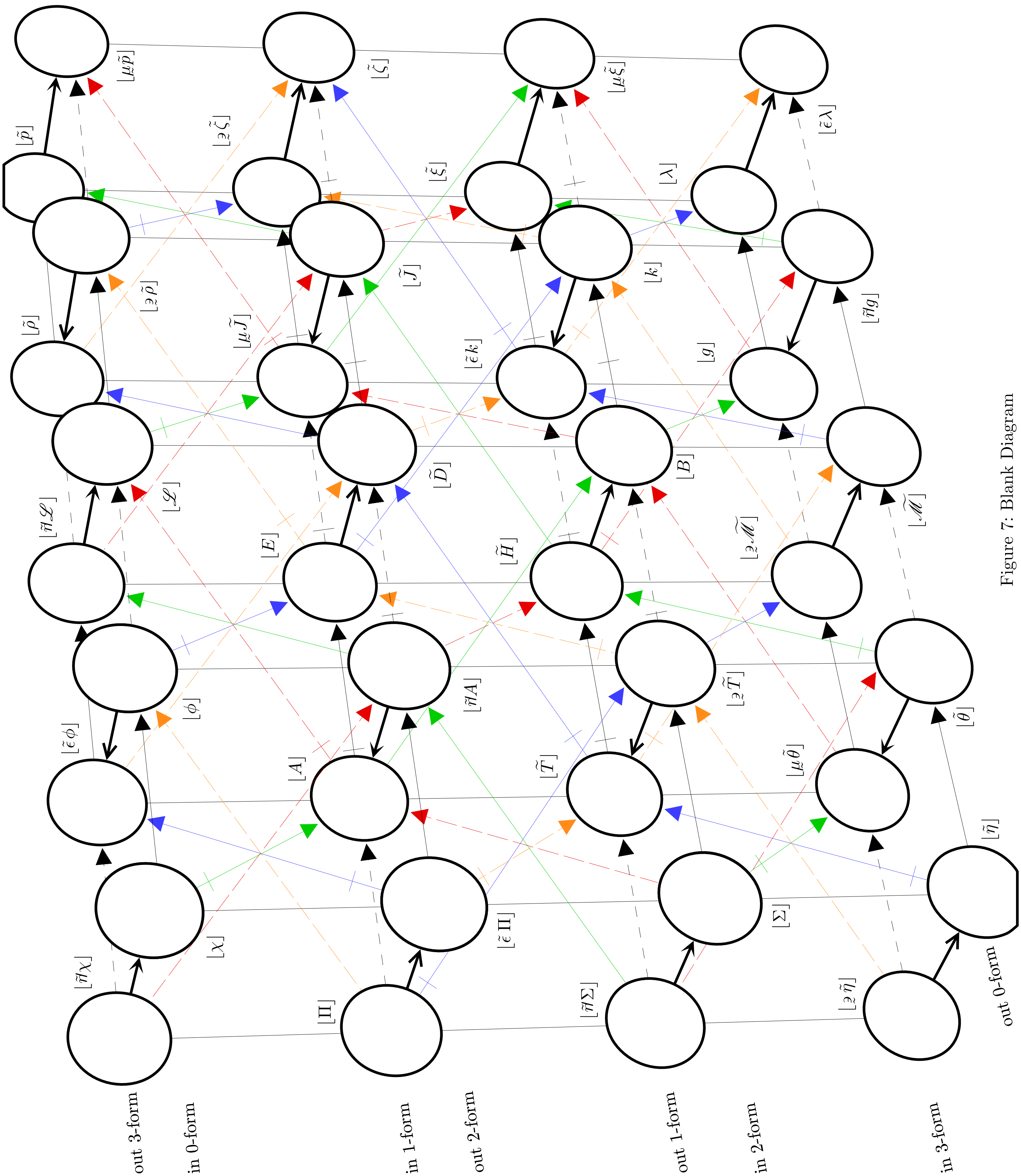

Figure 7: Blank Diagram

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